Labor Responses, Regulation and Business Churn in a Small Open Economy

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Abstract

We analyze labor responses to technology shocks when firm entry is sluggish due to endogenous sunk costs. We provide closed-form solutions for transition dynamics that show, when firm entry is slow to respond, labor will increase (decrease) relative to its long-run response if returns to labor input at the firm level are increasing (decreasing). Under stricter regulation (slower business churn), such short-run deviations of labor persist for longer. There is also potential for short-run productivity effects to differ from the long run.

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1 Introduction

A long-standing debate in business cycle theory is whether aggregate labor (hours) increases or decreases on impact of a technology shock. We reconcile both labor responses through two observations: (i) returns to labor inputs are not constant at the firm level (ii) firm entry is not instantaneous. Specifically, we argue that sluggish firm entry adjustment affects labor per firm which, through returns to scale in labor input, determines short-run responses. And regulation (specifically red tape) slows the rate of business churn increasing the persistence of short-run labor deviations.

There is a growing body of literature that considers the aggregate implications of firm entry and exit dynamics at a business cycle frequency.[1] Most of this literature assumes *internal* returns to scale in input factors are constant (or restricted to be decreasing as in Hopenhayn 1992).[2] However with a fixed overhead cost and imperfect competition, restrictions on returns to scale in labor input are unnecessary: we derive existence conditions that allow us to study arbitrary returns to labor input. Our contribution is to explain that the response of aggregate labor to technology shocks depends on whether slow firm entry increases or decreases labor's efficiency at the firm level. And, regardless of impact effect, the speed of convergence depends on the flow of firm creation, which in turn depends on the level of regulation in an industry/economy. We characterize deregulation as a cut in red tape, which causes less congestion in the entry process decreasing the endogenous sunk entry cost and speeding-up business churn.

The intuition is that the number of firms in an industry is a quasi-input in production which represents the organization of the industry/economy. Ceteris paribus, an extra firm decreases labor *per firm* which is an input in the firm's production function. Under the common assumption of constant returns to firm-level labor, the decrease in labor per firm does not affect the

[1] Wang and Wen 2011; Bilbiie, Ghironi, and Melitz 2012; Bergin, Feng, and Lin 2016; Clementi and Palazzo 2016; Sedlacek and Sterk 2017; Carvalho and Grassi 2017.

[2] Emphasis on *in factors* as opposed to returns to scale from fixed costs, and *internal* as opposed to external returns to scale aggregation, which is studied in an entry context by Barseghyan and DiCecio 2016.

marginal product of labor (MPL), wages or in turn aggregate labor. But if returns to labor are decreasing, the decline in labor per firm increases MPL, which in turn increases wages and thus aggregate labor. The opposite holds with increasing returns at the firm level.

Basu, Fernald, and Kimball 2006 show that in US manufacturing industries (durable and non-durable) returns to labor (hour per worker) are increasing, whereas in non-manufacturing are decreasing. For this example, our theory implies that a technology shock to manufacturing initially increases labor and wages relative to their long-run level, but then subsequent entry of new firms decreases wages and labor; whereas in non-manufacturing a positive technology shock would initially have an undershooting effect on labor and wages, then as firms enter wages and hours increase to their new long-run level. More recently Cantore, Ferroni, and Leon-Ledesma 2017 (Fig. 1, p.70) provide empirical evidence that, at an aggregate level, the response of labor to technology shocks has reversed over the past century in the US, and that the deviation now persists for longer. We find that increased persistence can be explained by greater regulation (more red tape) that slows business churn.[3] Data reported in Figure 1 indicate that red tape, proxied by procedures to start a business, is positively related to the length of time it takes to start a firm which proxies pace of business formation.[4]

We develop a general equilibrium firm dynamics model for a small open economy (SOE) in continuous time. Households are able to perfectly smooth consumption, and they can invest in setting up new firms by paying an endogenous sunk entry cost. Once operational, firms compete under monopolistic competition, and pay a fixed overhead cost each period. The SOE setup eliminates other sources of dynamics so that we can focus on firm entry dynamics. There is no capital, only labor, and there is an internationally traded bond with world interest rates equal to the household discount

[3]This speaks to recent literature on *'declining business dynamism'* (Decker et al. 2018) that links 'declines in the pace of business formation' to slower reallocation of resources.

[4]Using the World Bank Doing Business database, the scatter in Figure 1 represents 211 countries, and a small number of economic zones, for 2016. Venezuela is the 20 procedures 230 days outlier, and New Zealand is the 0.5 days 1 procedure point. Ebell and Haefke 2009 report similar trends in number of procedures and days to start-up for OECD data.

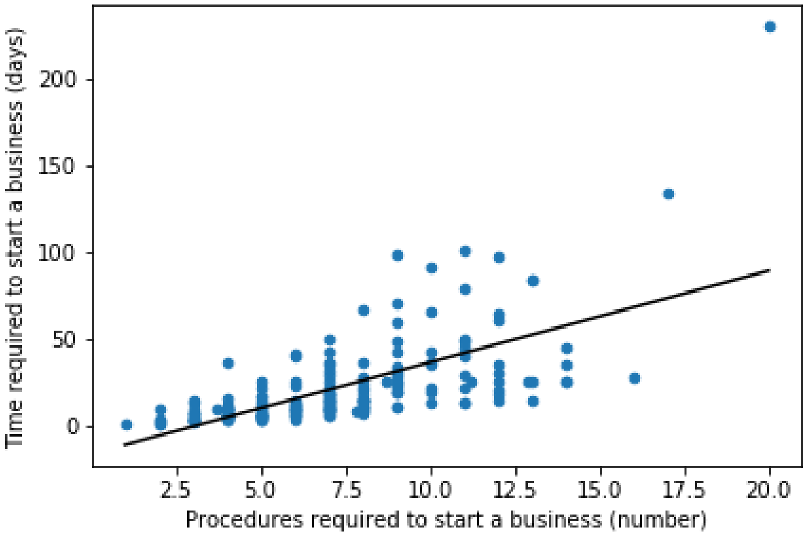

Figure 1: Red Tape and Business Churn

rate. This implies the household is able to perfectly smooth utility, so consumption dynamics do not play a role. The restriction to one state variable (number of firms), keeps eigenvalues tractable, so we can study speeds of adjustment (Chatterjee 2005) and short-run versus long-run effects analytically. To model dynamic entry we assume that the entry cost depends on the flow of entry due to a congestion effect caused by red tape (Datta and Dixon 2002).[5] There is an intertemporal zero-profit condition which equates the cost of entry in each instant to the net present value of incumbency, resulting in the number of firms gradually adjusting towards its long-run value. Output per firm and operating profits vary in the short run, whilst in the long-run firms fully adjust so that there is a free-entry, zero-profit, steady-state characterized by an average firm size that is unaffected by technology change.[6]

To gain insight into the mechanism of adjustment, and associated labor

[5]This form of endogenous sunk cost follows a line of research established by Sutton 1991, and pursued primarily in IO work (Ericson and Pakes 1995; Das and Das 1997), although endogenous sunk costs are gaining attention in macroeconomics (Lewis 2009; Bergin and Lin 2012; Berentsen and Waller 2015).

[6]This dynamic entry model has both instantaneous free-entry (Jaimovich 2007; Jaimovich and Floetotto 2008) and no-entry, fixed firms as limiting cases.

responses, consider a positive permanent technology shock. Initially firms are in a long-run steady-state position with zero profits. This free-entry outcome occurs because past entry has adjusted to reduce demand to each firm such that its output, which is sold at a markup, generates just enough revenue to cover fixed costs.[7] At this steady-state point firms always have increasing returns to scale from overhead costs, regardless of returns to scale in labor (marginal costs) which can be increasing, decreasing or constant. When the permanent technology improvement occurs, *per firm* operating profits (dividends), and analogously output *per firm*, increase as technology directly improves the production technology, and indirectly affects labor supply which adjusts instantaneously.[8] Entry (a flow) also responds instantaneously in response to positive profit, but this only affects the number of firms (a stock) an instance later. Therefore variations in labor *per firm* are solely driven by changes in aggregate labor on impact, and this variable will evolve over time as the number of firms begins to adjust. Subsequently as new firms enter the economy, whether entry increases, decreases or has no effect on aggregate labor will depend on the returns to scale of labor in a firm's production function.

Additionally the short-run, procyclical movements in output per firm (firm scale) and profits create procylical movements in measured productivity as firms temporarily exploit increasing returns to scale from the fixed cost. This is a Hall 1990 interpretation of endogenous productivity movements: firms move down their average cost curve as capacity increases. However, here it is *temporary* due to subsequent entry responses alleviating incumbents' initial capacity response.

Our model is supported by several stylized facts[9] and contributes to the

[7]Endogenous sunk costs are zero in steady state.

[8]Labor supply moves ambiguously depending on income and substitution effects. Our general formulation will show that labor per firm will not necessarily increase, but output per firm must increase so profits increase.

[9]Procylical net entry which lags the cycle (Campbell 1998; Bergin, Feng, and Lin 2016); the existence of variable returns to scale in labor (Basu, Fernald, and Kimball 2006); the existence of monopoly power (De Loecker and Eeckhout 2017); procylical average firm scale (capacity utilization) which is contemporaneous with the cycle; procylical firm profits also contemporaneous with the cycle (Lewis 2009; Bilbiie, Ghironi, and Melitz 2012); countercylical labor share (Young 2004); and procylical measured productivity (Basu and

debate on labor responses to technology shocks (Gali 1999; Basu, Fernald, and Kimball 2006; Christiano, Eichenbaum, and Vigfusson 2004; Wang and Wen 2011; Rebei 2014). Our endogenous sunk cost approach differs from current models of endogenous dynamic entry that assume fixed sunk costs and a time-to-build lag in discrete time (Bilbiie, Ghironi, and Melitz 2012; Lewis and Winkler 2017). Rather than modelling regulation through variation in exogenous sunk costs (Bilbiie, Ghironi, and Melitz 2007; Poschke 2010; Barseghyan and DiCecio 2011; Storesletten, Kambourov, and Brandt 2016; Cacciatore and Fiori 2016; Asturias et al. 2017), which typically addresses long-run outcomes, we look at deregulation that reduces endogenous sunk costs and focus on dynamic convergence.[10] Our model adds a new dimension to the analysis of labor responses to technology (or fiscal) shocks by providing a theoretical framework which shows how both the behaviour of the marginal product of labor and the response of entry costs to the flow of entry are crucial ingredients determining short and long-run behavior.

The remainder of the paper is as follows: Section 2 outlines the household's problem; Section 3 analyzes the firm's production problem and endogenous entry decision; Section 4 collects together the equilibrium conditions, then solves for steady-state and transition paths; Section 5 analyzes labor responses and Section 6 shows that deregulation speeds-up convergence.

2 Household

There is a small open economy, with a world capital market interest rate r equal to the discount rate ρ of the Ramsey household.[11]

$$r = \rho \tag{1}$$

Fernald 2001; Jaimovich and Floetotto 2008).

[10]As clarified earlier, less stringent regulation reduces congestion in firms' entry thereby decreasing endogenous sunk costs. Whereas the literature that focuses on 'deregulating' fixed sunk costs is primarily concerned with the absolute number of firms that operate in steady state, our interest is on the speed at which the number of firms adjusts to steady state.

[11]This so-called knife-edge condition is a widely-discussed, model closing device (Turnovsky 2002; Oxborrow and Turnovsky 2017).

We assume King-Plosser-Rebelo preferences with logarithmic consumption utility.

$$U(C, 1-H) = \ln C - \frac{H^{1+\eta}}{1+\eta} \tag{2}$$

$\eta = \frac{1}{FE} \in (0, \infty)$ is inverse Frisch elasticity of labor supply to wages. We ignore indivisible labor $\eta = 0$. The instantaneous utility function $U(C, 1-H)$ is twice continuously differentiable and strictly concave with utility increasing in consumption and decreasing in labor. Additive separability in labor and consumption $U_{CH} = 0$ implies that the marginal utility of consumption is independent from labor.[12] H lies in a convex compact set $H \in [0, \bar{H}]$.

The household earns income from three sources: supplying labor at wage w, receiving interest income from net foreign bonds rB and receiving profit income Π from owning firms. The household treats profit income as a lump sum payment. The government finances its expenditure G by a lump-sum tax equal to expenditure in each instant.[13] The household solves:

$$\max \int_0^\infty U(C, H) e^{-\rho t} dt \tag{3}$$

$$\text{subject to} \quad \dot{B} = rB + wH + \Pi - C - G \tag{4}$$

$$B(0) = B_0 \tag{5}$$

$$\text{where} \quad r = \rho \tag{1}$$

Given KPR preferences the optimal solutions satisfy

$$\dot{\lambda} = 0 \implies \lambda = \bar{\lambda} \tag{6}$$

$$\bar{C} = \frac{1}{\bar{\lambda}} \tag{7}$$

$$H^F(w, \lambda) = (\lambda w)^{\frac{1}{\eta}}, \quad \eta \in (0, \infty) \tag{8}$$

[12] Additive separability $U_{CH} = 0$ is sufficient for our results to hold when there are increasing marginal costs (decreasing returns to labor). But we require KPR preferences for the decreasing and constant marginal cost cases.

[13] This is for convenience and avoids the need for introducing government bonds. Since Ricardian equivalence holds, the timing of taxation does not matter. We include this for the sake of our supplementary results which show that our technology shock results also hold for government spending shocks.

where we use bar notation for variables that are constant over time. For a given wage, Frisch elasticity of labor supply H^F is increasing in λ. Frisch elasticity of supply measures the substitution effect of a change in the wage rate on labor supply $H^F_w \frac{w}{H^F} = \frac{1}{\eta}$.[14] The perfect capital markets assumption $r = \rho$ (implies constant consumption $\dot{\lambda} = 0$) and additively separable utility U_{CH} are responsible for giving these simple dynamics.[15] The result is that the only dynamics in the model will be a result of firm entry, which will affect wage. The advantage is to pinpoint the precise role of firm entry. Since λ represents the marginal utility of consumption: a high λ means a low level of consumption and vice versa.

Lastly, to ensure the private agent satisfies the intertemporal budget constraint, the transversality condition must hold

$$\lim_{t \to \infty} \lambda B e^{-rt} = 0 \tag{9}$$

Therefore the solution to the problem is characterized by two boundary conditions (5), (9) and two ordinary differential equations (ODEs) $\dot{\lambda}, \dot{B}$ that solve to give trajectories $B(t), \lambda(t)\ \forall t$. Subsequently $\lambda(t)$ gives $C(t)$ and in turn $H(t)$ through the static conditions. However before solving we need to characterize w and Π behave endogenously in general equilibrium according to factor market equilibrium that we develop next.

[14] See the appendix for full derivation of first-order conditions. We rule out indivisible labor $\eta = 0$ which would imply $C = w$. If $r \neq \rho$ then no interior steady state exists. The trajectory of consumption will then be either increasing $r > \rho$ or decreasing $r < \rho$ through time. There are many discussions of *'closing devices'* (or *'stationarity-inducing devices'*) in the SOE literature, which are necessary because the exogenous world interest rate causes an incomplete market. See Seoane 2015 based on Mendoza 1991. Oxborrow and Turnovsky 2017 give overview and close the model using demography.

[15] Additive separability $u_{CH} = 0$ creates the simple relationship between consumption and marginal utility of consumption. The presence of a small open economy and perfect international capital markets $\rho = r$ implies the household can completely smooth its consumption so $\dot{\lambda} = 0 \implies \lambda = \bar{\lambda}$. Therefore together they imply the marginal utility of consumption is unchanging over time.

3 Firms: Technology, Entry and Exit

The aggregate consumption good C is either imported or produced domestically by a perfectly competitive industry with a $CRTS$ production function using intermediate inputs which are monopolistically supplied. There is a continuum of possible intermediate products, $i \in [0, \infty)$. At instant t, there is a range of active products defined by $N(t) < \infty$, so that $i \in [0, N(t))$ are *active* and available, whilst $i > N(t)$ are inactive and not produced.

Total domestic output Y is related to inputs y_i by the following technology

$$Y = N^{\varsigma - \frac{\theta}{\theta-1}} \left[\int_0^N y_i^{(\theta-1)/\theta} di \right]^{\theta/(\theta-1)} \tag{10}$$

where $\theta > 1$ is the elasticity of substitution between products. The N^{ς} multiplier captures any *Ethier effect*, we remove this by assuming $\varsigma = 1$ so an increase in the range of intermediates does not affect the unit cost function.[16] Treating the unit price of the consumption good as the numeraire, the demand for each available product i takes the constant elasticity form

$$y_i = p_i^{-\theta} \frac{Y}{N^{\varsigma}} \tag{11}$$

There is a continuum of potential firms: each firm can produce only one product. Therefore the index of firms and products is the same. At time t, firm $i \in [0, N(t))$ has labor demand h_i to supply output y_i using the technology

$$y_i = A h_i^{\nu} - \phi \tag{12}$$

where $\nu > 0$ captures labor returns to scale ($\nu < 1$ decreasing returns; $\nu = 1$ constant returns; $\nu > 1$ increasing returns), and $\phi \geq 0$ is a fixed flow overhead and $A > 0$ is a technology parameter which increases the marginal product of labor.[17] When $\nu < 1$, $\phi > 0$ there is an *U-shaped* average cost

[16] A common case is $\varsigma = 0$ which leads to a variety effect, we want to remove this as it will create an additional mechanism adding to the main result we want to distill. Without removing love of variety, N will enter the labor market equilibrium condition, even with constant returns to scale.

[17] In this paper overheads represent foregone output as in Jaimovich and Floetotto 2008;

(AC) with increasing marginal cost (MC); this is compatible with both perfect (Walrasian) and imperfect competition. When $\nu = 1, \phi = 0$, we have constant returns to scale: $AC = MC$. When $\nu = 1, \phi > 0$, we have constant MC and decreasing AC; when $\nu > 1$ there is decreasing AC and MC (the extent to which ν can exceed 1 is limited - the upper bounds are stated below). In these last two cases with globally increasing returns to scale, equilibrium can only exist with imperfect competition.

3.1 Aggregate Output

The number of active firms at instant t is denoted $N(t)$: we will drop the time index when it does not lead to ambiguity. Throughout we will be assuming that labor markets function perfectly so that labor is allocated equally across firms, so that $h_i = H/N$, $\forall i \in N$. The *aggregate production function*, obtained from (10), (12) under symmetry is homogeneous of degree 1 in N and H

$$Y(N,H) = AH^{\nu}N^{(1-\nu)} - N\phi \tag{13}$$

Treating N, H as independent, the partial derivatives are[18]

$$Y_N \equiv \frac{\partial Y}{\partial N} = (1-\nu)Ah^{\nu} - \phi = y - \nu Ah^{\nu} = (1-\nu)y - \nu\phi \gtrless 0 \tag{14}$$

$$Y_H \equiv \frac{dY}{dH} = A\nu(H/N)^{\nu-1} = A\nu h^{\nu-1} = \nu\frac{y+\phi}{h} > 0 \tag{15}$$

When there are increasing returns to labor $\nu > 1$, an additional firm can decrease aggregate output as it employs labor less productively than the incumbents did prior to its entry. Since Y is homogeneous of degree 1 in $\{N, H\}$ then by Euler's homogeneous function theorem

$$Y = Y_N N + Y_H H \tag{16}$$

Devereux, Head, and Lapham 1996. An alternative is to have the overhead in terms of labor.

[18] It is important to note the N derivative is partial, as the in general equilibrium the total derivative would recognize that a variation in N implicitly varies H, that is $\frac{dY}{dN} = \frac{\partial Y}{\partial N} + \frac{dY}{dH}\frac{dH}{dN}$. Since N is independent of H then its partial and total derivative are equivalent.

The aggregate marginal product of labor equals the firm-level marginal product $Y_H = y_h$ because labor is allocated equally across firms. Later this implies that wage corresponds to either aggregate or firm level MPL.

3.2 Profits and Factor Market Equilibrium

In this section, we determine the operating profits of an active firm (a firm that does not incur any entry costs). Due to imperfect competition, the firm maximizes operating profits given real wage w and aggregate output Y (using output price as the numeraire $P = 1$) by choosing employment to satisfy the factor market equilibrium[19]

$$w = \frac{1}{\mu} Y_H = \frac{\nu}{\mu} A \left(\frac{H}{N} \right)^{\nu - 1} = \frac{\nu}{\mu} \frac{y + \phi}{h} \tag{17}$$

Where $\mu = \frac{\theta}{\theta - 1} \in [1, \infty)$ is the markup, which is 1 with perfect competition when products are perfectly substitutable $\theta \to \infty$, so demand curves are perfectly elastic.[20]

Lemma 1. *$\nu < \mu$ is a sufficient condition for the second-order profit maximization condition to hold. It is necessary and sufficient for steady-state existence.*

The aggregate flow of operating profits given w equals $N\pi$, where π is

[19]The result follows from the profit maximization problem outlined in Appendix A.4. In the increasing returns case $\nu > 1$, the second-order condition for profit maximization is not always satisfied, so we give a necessary condition for this. However, our later condition $\nu < \mu$ is sufficient for this second-order necessary condition to hold.

[20]Labor demand h will vary depending on returns to scale. The relationship captures *'aggregate labor demand'* (Jaimovich 2007), the right-hand side is the marginal revenue product of labor which is the inverse of the markup multiplied by the marginal product of labor. The number of firms affects the relationship through the marginal product of labor since the markup is fixed. With endogenous markups and constant returns to scale, the number of firms also affect the MRPL (also true of LOV). Both can create upward sloping marginal product schedule $dw/dH > 0$.

firm level profit.[21]

$$\pi = Y_N + \left(1 - \frac{1}{\mu}\right) Y_H \frac{H}{N} \tag{18}$$

Operating profits and output per firm (thus labor per firm) are isomorphic since $\pi = y - \frac{\nu}{\mu}(y + \phi)$ hence

$$\pi = y\left(1 - \frac{\nu}{\mu}\right) - \frac{\nu}{\mu}\phi = Ah^{\nu}\left(1 - \frac{\nu}{\mu}\right) - \phi \tag{19}$$

$$y = \frac{\mu\pi + \nu\phi}{\mu - \nu} \tag{20}$$

$$h = \left(\frac{y + \phi}{A}\right)^{\frac{1}{\nu}} = \left(\frac{\mu(\pi + \phi)}{A\,(\mu - \nu)}\right)^{\frac{1}{\nu}} \tag{21}$$

Operating profits respond less than one-for-one when output changes $\pi_y = 1 - \frac{\nu}{\mu} < 1$. The implication is that economic profits are less volatile than output, and lemma 1 implies that this relationship cannot be negative. Furthermore it formalizes that perfect competition $\mu = 1$ is possible if returns to scale are decreasing $\nu < 1$ (increasing marginal costs). This is important in steady state, which corresponds to zero profits $\pi = 0$, because it ensures non-negative steady-state output. It is a steady-state existence condition. This restriction implies that for profit maximizing output MR must intersect MC from above. A higher μ implies steeper MR which allows steeper downward sloping MC. Hence a higher degree of monopoly (more differentiated products) allows for larger ν, whereas horizontal MC can only exist if MR is downward sloping, so some monopoly power exists. Increasing marginal costs $\nu < 1$ is compatible with any level of imperfect competition $\mu \in [1, \infty)$. If $\nu > 1$ marginal cost is downward sloping, and some market power must exist (there is a limit on how low markups can go) $\mu > \nu$.

[21] The result follows from substituting w (17) and Y (16) out of the aggregate profit expression $N\pi = Y - wH$ such that $N\pi = Y_N N + Y_H H - \frac{Y_H}{\mu} H$, which rearranges to (18).

3.3 Productivity

This paper has dynamic entry so that $\pi = 0$ is not instantaneous, rather $\pi(t)$ decreases to zero as firms 'slowly' enter to arbitrage these profits. Unlike papers of static entry in which the zero-profit condition holds instantaneously as firms immediately arbitrage positive profits, here it is important to understand how these varying profits affect economic variables.

Since profits are not zero instantaneously then during this short-run period changes in output consist of changes in profits and changes in inputs (labor) $y = \pi + wh$. Therefore the wage share is countercyclical $\frac{wh}{y} = \frac{\nu}{\mu}\left(1 + \frac{\phi}{y}\right)$ and the profit share procylical $\frac{\pi}{y} = 1 - \frac{\nu}{\mu}\left(1 + \frac{\phi}{y}\right)$ as they perfectly crowd each other out. This implies that variation in output in the short-run consists of more than just variation in inputs, so productivity (output change explained by inputs) will appear higher than the case with zero profits as there is an extra positive term in the productivity numerator.

From (20) and (21) output per firm and labor per firm are increasing in operating profits

$$y_\pi = \frac{\mu}{\mu - \nu} > 0 \tag{22}$$

$$h_\pi = \frac{\mu}{A\nu(\mu - \nu)} h^{1-\nu} > 0 \tag{23}$$

Since $n\pi = Y - n\frac{\nu}{\mu}h^\nu$ then substitute out $n = \frac{Y}{y}$ gives aggregate output as a function of inputs and profits

$$Y = \frac{A^{\frac{1}{\nu}}(\mu - \nu)^{\frac{1-\nu}{\nu}}(\mu\pi + \nu\phi)}{(\mu(\pi + \phi))^{\frac{1}{\nu}}} H \tag{24}$$

We therefore define average labor productivity:[22]

$$\mathcal{P} = \frac{Y}{H} = \left(\frac{A(\mu-\nu)^{1-\nu}}{\mu}\right)^{\frac{1}{\nu}} \frac{(\mu\pi+\nu\phi)}{(\pi+\phi)^{\frac{1}{\nu}}} \tag{25}$$

3.4 Labor at Factor Market Equilibrium

In general for $\eta \in (0,\infty)$ and $\nu \in (0,\infty)$, the intratemporal (labor supply) condition (8) $w = H^\eta C$ and the wage market equilibrium (marginal product schedule) (17) $w = \frac{A\nu}{\mu} H^{\nu-1} N^{1-\nu}$ equate $H(\lambda, N, A) = H^F\left(\lambda, \frac{1}{\mu} Y_H(H,N)\right)$ to give $H^\eta C = \frac{A\nu}{\mu} H^{\nu-1} N^{1-\nu}$.[23] It is useful to write this as a function of (N,λ).

$$H(\lambda, N) = \left(N^{1-\nu}\lambda \frac{\nu A}{\mu}\right)^{\frac{1}{1+\eta-\nu}}, \quad 1+\eta-\nu > 0 \tag{26}$$

Lemma 2 (Labor Market Equilibrium Existence)**.** *To ensure that the labor market condition is well-defined* $\nu < 1+\eta$

The restriction $\nu < 1+\eta$ implies that the slope of the labor supply curve exceeds the slope of the labor demand curve. The labor supply curve $w = CH^\eta$ slope is $\frac{dw}{dH} = \frac{\eta w}{H}$, and upward sloping in (H, w) space (or flat with indivisible labor $\eta = 0$). This must be greater than the slope of the marginal (revenue) product schedule $w = \frac{A\nu}{\mu} h^{\nu-1}$, which is $\frac{dw}{dH} = w_h h_H = \frac{(\nu-1)w}{h}\frac{1}{N} = \frac{(\nu-1)w}{H}$, which reflects that demand for labor can be upward sloping if it has increasing returns to scale.

As usual we can interpret the result in terms of profits. We can define labor in terms of profits, and show its ambiguous response depends on returns

[22] An alternative representation of (25), which makes clear the role of returns to scale ν is $\mathcal{P} = \left(\frac{A}{\pi+\phi}\right)^{\frac{1}{\nu}} \left(1-\frac{\nu}{\mu}\right)^{\frac{1}{\nu}-1} \left(\frac{\nu\phi}{\mu}+\pi\right)$. With $\nu = 1$ this is equivalent to Jaimovich and Floetotto 2008, Supplementary Appendix eq. 17. They develop a similar firm structure but with static (instantaneous) firm entry.

[23] If labor is indivisible ($\eta = 0$) then all wage is consumed $C = \frac{1}{\mu} A\nu h^{\nu-1}$, so there is no substitution effect. With constant marginal costs $\nu = 1$ then $C = A/(\mu H^\eta)$ there is only an income effect as wage is fixed. Jaimovich 2007 studies the effect of instantaneous entry on this relationship with both constant returns and indivisible labor, but N affects the relationship through endogenous markups $\mu(N)$ which causes indeterminacy.

to scale. If we substitute out $N = H/h = H\left(\frac{A(\mu-\nu)}{\mu(\pi+\phi)}\right)^{\frac{1}{\nu}}$ in (26) we get

$$H = \left[\left(\frac{A(\mu-\nu)}{\mu(\pi+\phi)}\right)^{\frac{1-\nu}{\nu}} \frac{\lambda\nu A}{\mu}\right]^{\frac{1}{\eta}} \tag{27}$$

Whether labor increases, decreases or does not respond to a change in profits depends on returns to scale ν.

Proposition 1 (Existence)**.** *Necessary and sufficient condition for existence*

$$\nu < \min\left[\mu, 1+\eta\right] \tag{28}$$

Proof. Combine profit existence Lemma 1 and labor market existence Lemma 2. □

A sufficient condition is that there are increasing marginal costs $\nu < 1$. Where $1+\eta > 1$ because we rule out indivisible labor $\eta = 0$.

Entry alters employment per firm which, through marginal costs, affects the efficiency of labor and thus the real wage it is paid. With a decreasing MPL, entry increases the real wage and hence labor supply; with increasing MPL the opposite holds.

Proposition 2 (General Equilibrium Labor Behavior)**.** *From the labor market equilibrium condition* (26)*, we can see that labor responses to entry are*

$$H_N > 0 \iff \nu \in (0,1) \tag{29}$$

$$H_N = 0 \iff \nu = 1 \tag{30}$$

$$H_N < 0 \iff \nu \in (1,\infty) \tag{31}$$

In deriving this result we show that labor elasticity to number of firms $\varepsilon \equiv H_N \frac{N}{H}$ is constant and bounded

$$\varepsilon = \frac{1-\nu}{1+\eta-\nu} \tag{32}$$

It is bounded by $\frac{-\eta}{1-\nu+\eta} < \varepsilon < 1$. The upper bound occurs with indivisible labor $\eta \to 0$. The lower bound follows from $\nu < 1+\eta$ so that (working right to left) $\frac{-\eta}{1-\nu+\eta} < \frac{1-(1+\eta)}{1-\nu+\eta} < \frac{1-\nu}{1-\nu+\eta} = \varepsilon$. If $\nu = 1$ then $\varepsilon = 0$. If $\nu < 1$ then $0 < \varepsilon < 1$. And if $\nu > 1$ then $-\infty < \varepsilon < 0$.[24]

3.4.1 Total Derivatives: Labor Effect Vs. Business Stealing

In section 3.1 we derived the partial derivatives of aggregate output with respect to labor $Y_H > 0$ and firms $Y_N \gtreqless 0$, assuming H and N were independent. Now that we have determined $H(\lambda, N)$ we can assess total derivatives of output with respect to entry by noting how labor changes endogenously. Understanding this mechanism is important for our results on the effect of entry on aggregate output to be derived later. The main point is that entry has an ambiguous effect on aggregate output if there are decreasing returns $\nu < 1$ so that $\varepsilon > 0$. This is because entry strengthens labor supply which can increase output. Whereas with constant or increasing returns $\nu \geq 1$ an entrant always decreases aggregate output.

$$\frac{dY}{dN} = y + N\frac{dy}{dN} = \varepsilon(1+\eta)Ah^{\nu} - \phi \tag{33}$$

The first equality states that an entrant contributes its own output y but has a *business stealing* (Mankiw and Whinston 1986) effect on the output of all other incumbents. In the appendix we show this business stealing effect is strictly negative $N\frac{dy}{dN} = \nu(y+\phi)(\varepsilon - 1) < 0$. The second equality of (33) makes clear the crucial role of firm level returns to scale. It reads that an entrant has a negative effect by bringing in an extra fixed cost, but it has another positive, negative or zero effect depending on the labor elasticity to entry ε. In terms of profits this can be written $\frac{dY}{dN} = Y_N + Y_H H_N = \pi - \left(1 - \frac{1}{\mu} - \varepsilon\right) Y_H h$ which is useful when we analyze zero-profit steady state.[25] The first term is the partial derivative effect of an entrant (14) which we have explained is ambiguous based on ν, and the second term is the labor response which is also ambiguous based on ν.

[24] See Appendix A.5 for proof.
[25] See Appendix for full derivation and discussion.

Since y and π are in a one-one relationship, the business stealing effect can also be interpreted as entrants diminishing profits, from (19) $\frac{d\pi}{dN} = \frac{dy}{dN}\left(1 - \frac{\nu}{\mu}\right) < 0$. In the dynamic analysis we shall use the expression for dividends with $H(\lambda, N)$ substituted out:

$$\pi(\lambda, N) = \left(\frac{A^{1+\eta}(\nu\lambda)^{\nu}}{\mu^{1+\eta}N^{\eta\nu}}\right)^{\frac{1}{1+\eta-\nu}} (\mu - \nu) - \phi \tag{34}$$

3.5 The Entry Decision

What determines the number of firms operating at each instant t? We develop a congestion effects model of firm entry such that at time t there is a flow cost of entry $q(t)$ which increases in net entry $E(t)$.[26]

$$E(t) \equiv \dot{N} \tag{35}$$

$$q(t) = \gamma E(t) \tag{36}$$

The sensitivity to congestion parameter $\gamma \in (0, \infty)$ represents red tape or regulation in firm creation. Filing papers or gaining accreditation makes start-ups more sensitive to flows of entry as regulator's offices become more congested (i.e. a queuing cost). Aggregating across all entry in a period gives a quadratic firm entry adjustment cost function

$$\mathcal{C}(E) \equiv \int_0^E q\, dE = \frac{\gamma}{2}E^2 = \frac{q^2}{2\gamma} \tag{37}$$

$\mathcal{C}(E)$ is a non-negative, convex function of the rate of entry. With zero entry, the aggregate cost and marginal cost of firm creation is zero $\mathcal{C}(0) = \mathcal{C}_E(0) = 0$. The interpretation of modelling the aggregate sunk cost as an adjustment cost is that firm creation and destruction, whether positive (net entry) or negative (net exit), generates resource costs.

[26] Entry and exit are symmetric, with $-q$ being the cost of exit at time t. There are sunk costs to entry and dismantling fees, such as severance payments, to exit. See Das and Das 1997; Datta and Dixon 2002 for further details. Exit and entry symmetry is not essential, exit could require a fixed cost, perhaps zero, as in Das and Das 1997 and Hopenhayn 1992 or evolve endogenously according to productivity Melitz 2003; Hamano and Zanetti 2017.

The flow of entry in each instant is determined by an *arbitrage condition* that equates the return on bonds (opportunity cost of entry) with the return on setting up a new firm. It is a differential equation in q, which determines the entry flow by (36).[27]

$$\frac{\pi}{q} + \frac{\dot{q}}{q} = r \tag{38}$$

π is given by (18) which will make this a nonlinear differential equation in N.[28] The first left-hand side term is the number of firms per dollar $(1/q)$ times the flow operating profits (dividends) the firm will make if it sets up. The second term reflects the change in the cost of entry. If $\dot{q}/q > 0$, then it means that the cost of entry is increasing, so that there is a capital gain associated with entry at time t if $\dot{q}/q < 0$ it means entry is becoming cheaper, thus discouraging immediate entry. The sunk cost $q(t)$ represents the net present value of incumbency: it is the present value of profits earned if you are an incumbent at time t.[29] This arises since the entrants are indifferent between entering and staying out. When $q < 0$, the present value of profits is negative: in equilibrium this is equal to the cost of exit. In steady state, we have $E = q = 0$, so that the entry model implies the zero-profit condition. Entry costs only arise on convergence to steady state.

Accounting for entry costs, aggregate profits Π are the operating profits (dividends) of firms less the entry costs paid by the entrants

$$\Pi = N\pi - \gamma\frac{E^2}{2} = NY_N + \left(1 - \frac{1}{\mu}\right) Y_H H - \gamma\frac{E^2}{2} \tag{39}$$

Alternatively $\Pi = Y(N, H) - wH - \frac{q^2}{2\gamma}$.

[27] The arbitrage equation can be written in a way directly analogous to the user cost of capital $\pi = q\left(r - \frac{\dot{q}}{q}\right)$ in capital adjustment cost models.

[28] Note that our entry model has the standard models as limiting cases: when $\gamma = 0$, we have instantaneous free entry so that (38) becomes $\pi = 0$ and there are zero profits each instant. If $\gamma \to +\infty$, then changes in N become very costly and N moves little if at all which approximates the case of a fixed number of firms.

[29] This is because of the free-entry assumption that sunk costs equal the net present value of the firm. See Stokey 2008 for a general discussion.

4 Equilibrium Conditions, Steady State and Solution

The economic system is five dimensional $\{\lambda, N, q, B, H\}$ with four differential equations and one static equation. The static intratemporal condition (26) implies $H(\lambda, N)$, so the system can be reduced to four differential equations in four unknowns, and since the consumption differential equation implies consumption is constant $\lambda(t) = \bar{\lambda}$, we have three dynamic equations in N, q, B.

$$\dot{\lambda} = 0 \implies \lambda(t) = \bar{\lambda}$$

$$\dot{N}(q) = \frac{q}{\gamma} \tag{40a}$$

$$\dot{q}(N, \bar{\lambda}, q) = rq - \pi(N, H(\bar{\lambda}, N)) \tag{40b}$$

$$\begin{aligned} \dot{B}(B, N, \bar{\lambda}, q) &= rB + wH(\bar{\lambda}, N) + \Pi(N, H(\bar{\lambda}, N), q) - \bar{C}(\bar{\lambda}) - G \\ &= rB + Y(N, H(\bar{\lambda}, N)) - \mathcal{C}(q) - \bar{C}(\bar{\lambda}) - G \end{aligned} \tag{40c}$$

Alongside the four differential equations in system (40) there are four boundary conditions: the household transversality (9); the number of firms transversality; the initial condition on bonds; the initial condition on number of firms. Notably the industry dynamics (N, q) form a two dimensional subsystem of the three dimensional system, with bonds being B determined through (40c) alone. Therefore we shall solve recursively: first solving the industry dynamics subsystem for $N(t), q(t)$, then solve for bonds $B(t)$ based on these solutions.

4.1 Steady-state

Steady state is non-standard because there are three steady state conditions $\dot{N} = \dot{q} = \dot{B} = 0$ but four unknowns $\bar{\lambda}, q, N, B$.[30] In order to get an extra equation to solve this system for steady state, first we find a solution to

[30] This occurs because the consumption differential equations is always in steady-state ($\dot{\lambda} = 0$) due to perfect consumption smoothing from $r = \rho$ which implies consumption is fixed $\lambda = \bar{\lambda}$, but it does not relate to other variables in the system.

the dynamic system for its timepaths of $N(t,\bar{\lambda}), q(t,\bar{\lambda}), B(t,\bar{\lambda})$ conditional on knowing one steady-state variable $\bar{\lambda}$. Second we use the limit of the bond solution and transversality to acquire an extra steady state condition, allowing us to solve for steady state. It is this procedure which causes steady state to depend on initial conditions N_0, B_0, so-called path dependency or hysteresis.[31]

We use a tilde to denote a steady state variable. The $\dot{N} = 0$ differential equation immediately implies that steady-state sunk costs are zero, which equivalently implies the net present value of a firm in steady state is zero.

$$\tilde{q} = 0 \tag{41}$$

This leaves two steady-state conditions $\dot{q} = \dot{B} = 0$ in three unknowns $\tilde{N}, \bar{\lambda}, \tilde{B}$. Through the arbitrage condition (40b), zero sunk costs (41) imply operating profits are zero

$$\tilde{\pi} = 0 \tag{42}$$

Zero operating profits implies $0 = \tilde{y} - \tilde{w}\tilde{h}$ so the labor share is 100% of output in steady state and steady-state productivity equals the steady-state wage

$$\tilde{\mathcal{P}} = \frac{\tilde{y}}{\tilde{h}} = \tilde{w} \tag{43}$$

There is a fixed long-run aggregate output to aggregate labor ratio (labor productivity). Hence free entry zero profits impose long-run constant-returns on the relation between $\tilde{H}$ and $\tilde{Y}$. Irrespective of the value of ν at the firm level, a rise in $\tilde{H}$ will increase aggregate output by a fixed proportion $\tilde{\mathcal{P}} = \tilde{w}$. The zero profit condition determines labor per firm, or aggregate labor as a linear function of number of firms $\tilde{H}(\tilde{N})$

$$\tilde{h} = \left(\frac{\mu\phi}{A(\mu - \nu)}\right)^{\frac{1}{\nu}} \tag{44}$$

[31] An implication of this feature is that temporary shocks may have permanent effects.

Labor per firm determines output per firm and labor productivity (which equals wage).

$$\tilde{y} = \frac{\nu}{\mu - \nu}\phi \tag{45}$$

$$\tilde{\mathcal{P}} = \tilde{w} = \left(\frac{A}{\mu}\right)^{\frac{1}{\nu}} \nu \left(\frac{\phi}{\mu - \nu}\right)^{1-\frac{1}{\nu}} \tag{46}$$

With $\tilde{h}$ and $\tilde{w}$ determined by the free entry arbitrage condition $\tilde{\pi} = 0$, then the labor market equilibrium condition (26) determines the number of firms as a function of the consumption index, and therefore labor as a function of consumption index:

$$\tilde{N}(\bar{\lambda}) = \frac{(\bar{\lambda}\tilde{w})^{\frac{1}{\eta}}}{\tilde{h}} \tag{47}$$

$$\tilde{H}(\bar{\lambda}) = (\bar{\lambda}\tilde{w})^{\frac{1}{\eta}} \tag{48}$$

In order to find $\bar{\lambda}$, we are left with one steady-state condition $\dot{B} = 0$ that we have not used: the output market clearing condition (steady-state bond accumulation equation).

$$G + \bar{C}(\bar{\lambda}) - \tilde{w}\tilde{H}(\bar{\lambda}) - r\tilde{B} = 0 \tag{49}$$

This is an excess demand function for the steady state in terms of the price of marginal utility $\bar{\lambda}$. The first two terms $G + C(\bar{\lambda})$ represent expenditure and are decreasing in $\bar{\lambda}$. The second two terms $wH(\bar{\lambda}) + r\tilde{B}$ represent income and are increasing in $\bar{\lambda}$. By the intermediate value theorem, this implies that there exists a $\bar{\lambda} > 0$ such that the economy is at the steady state equilibrium given $\tilde{B}$ (See Appendix A.7 for proof of existence and uniqueness with endogenous $\tilde{B}(\bar{\lambda})$.).

In this section we partly defined steady-state $\{\tilde{N}, \bar{\lambda}, \tilde{B}\}$ for the primitive variables of the dynamical system $N, \bar{\lambda}, B$, given steady-state bonds $\tilde{B}$. We gave $\tilde{N}(\bar{\lambda})$ analytically in (47), then used (49) to prove a steady-state $\bar{\lambda}$ must exist given $\tilde{B}$. In the next section, we derive solutions for dynamics which provide an additional steady-state condition $\tilde{B}(\bar{\lambda})$ that teamed with (49) and

(47) can solve for $\bar{\lambda}$ by expressing (49) entirely in $\bar{\lambda}$ terms

$$G + \frac{1}{\bar{\lambda}} - \tilde{w}^{1+\frac{1}{\eta}}\bar{\lambda} - r\tilde{B}(\bar{\lambda}) = 0$$

4.2 Linearized system

The analysis of the steady state was conditional on the level of steady state bonds $\tilde{B}$. However to determine $\tilde{B}$ we need to know the path taken to equilibrium. The dynamics of the system will be analyzed by linearizing around the steady state. The Jacobian matrix of the 3-dimensional system is as follows (all elements are evaluated at steady state)

$$\mathbf{J} = \left[\begin{array}{ccc} 0 & \frac{d\dot{N}}{dq} & 0 \\ \frac{d\dot{q}}{dN} & \frac{d\dot{q}}{dq} & 0 \\ \frac{d\dot{B}}{dN} & \frac{d\dot{B}}{dq} & \frac{d\dot{B}}{dB} \end{array}\right]\Bigg|^{\tilde{}} = \left[\begin{array}{ccc} 0 & \frac{1}{\gamma} & 0 \\ -\frac{\tilde{d\pi}}{dN} & r & 0 \\ \frac{\tilde{dY}}{dN} & -\frac{\tilde{d\mathcal{C}}}{dq} & r \end{array}\right] \tag{50}$$

$$\frac{\tilde{d\mathcal{C}}}{dq} = \frac{\tilde{q}}{\gamma} \tag{51}$$

$$\frac{\tilde{d\pi}}{dN} = \frac{\tilde{\pi} + \phi}{\tilde{N}(\bar{\lambda})}\left(\frac{-\eta\nu}{1+\eta-\nu}\right) \tag{52}$$

$$\frac{\tilde{dY}}{dN} = A\tilde{h}^{\nu}\left(1 + \nu\left(\frac{1-\tilde{h}}{\tilde{h}}\right)\right) - \phi \tag{53}$$

where $\tilde{q} = \tilde{\pi} = 0$ (from (41) and (42)) and (44) gives $\tilde{h}$ as a function of exogenous parameters, but $\tilde{N}(\bar{\lambda})$ depends on endogenously determined steady-state consumption index given in (47). Section 3.4.1 and equation (34) help with these derivations, and make clear that both π an Y responses depend on business stealing $\frac{\tilde{dy}}{dN}$. Since the total effect of an entrant on aggregate output is an important mechanism for our analysis we shall denote it

$$\Omega \equiv \frac{dY}{dN}$$

The ambiguous effect of entry on aggregate output ($\Omega \gtreqless 0$) explored away from steady state in section 3.4.1 is also ambiguous in steady state ($\tilde{\Omega} \gtreqless 0$).

It depends on $\left(\varepsilon - 1 + \frac{1}{\mu}\right)$. We discuss this extensively in section 4.2.3. For dynamics it implies that the effect of entry on bond accumulation $\frac{d\dot{B}}{dN}|^{\tilde{}}$ is ambiguous. The linearized system can be written

$$\begin{bmatrix} \dot{N} \\ \dot{q} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\gamma} & 0 \\ \frac{1}{\tilde{N}(\bar{\lambda})} \frac{\nu\eta\phi}{1+\eta-\nu} & r & 0 \\ \tilde{\Omega} & 0 & r \end{bmatrix} \begin{bmatrix} N(t) - \tilde{N} \\ q(t) - \tilde{q} \\ B(t) - \tilde{B} \end{bmatrix} \tag{54}$$

$$\text{where} \quad \tilde{\Omega} = \frac{\nu\phi\mu}{\mu - \nu}\left(\varepsilon - 1 + \frac{1}{\mu}\right) \tag{55}$$

4.2.1 Number of Firms and Entry (industry dynamics) Solution

The determinant and trace of the industry dynamics $\{N, q\}$ sub-system $\mathbf{B} \in \mathbb{R}^2$ in (54) are

$$\det(\mathbf{B}) = \Delta = \frac{\frac{d\tilde{\pi}}{dN}}{\gamma} = -\frac{\nu\eta\phi}{\gamma\,(1+\eta-\nu)\,\tilde{N}(\bar{\lambda})} < 0 \tag{56}$$

$$tr(\mathbf{B}) = r \tag{57}$$

$\det(\mathbf{B})$ is negative as $1 + \eta > \nu$ and is increasing in $\bar{\lambda}$.[32] The root to the characteristic polynomial corresponding to the subsystem is

$$\Gamma(\bar{\lambda}) = \frac{r}{2}\left(1 \pm \frac{1}{r}\left[r^2 - 4\Delta(\tilde{N}(\bar{\lambda}))\right]^{\frac{1}{2}}\right) \tag{58}$$

The discriminant (square root term) is positive since the determinant is negative ($\Delta < 0$). This implies two distinct real roots. And since the discriminant exceeds 1, then so does its square root so there will be one positive and one negative root. Hence the system is saddle-path stable, with a negative real root Γ and a positive real root Γ^U. Furthermore the trace is positive so the sum of the eigenvalues is positive implying the positive eigenvalue is larger than the absolute value of the negative eigenvalue. Our focus is the stable

[32] See Appendix A.6 for proof.

root which is negative

$$\Gamma = \frac{1}{2}\left(r - \left[r^2 - 4\Delta)\right]^{\frac{1}{2}}\right) \tag{59}$$

Lemma 3. *The stable eigenvalue is increasing in* $\bar{\lambda}$

$$\Gamma_{\bar{\lambda}} = -\frac{\Delta}{\eta\bar{\lambda}(r^2 - 4\Delta)^{\frac{1}{2}}} = \frac{1}{\eta\bar{\lambda}}\frac{\Gamma(\Gamma - r)}{r - 2\Gamma} > 0$$

Proof. See Appendix A.6. □

The solution to the linearized subsytem is

$$N(t) = \tilde{N} + \exp[\Gamma(\bar{\lambda})t](N_0 - \tilde{N}) \tag{60}$$

take derivative to get the net entry rate $E = \dot{N} = \Gamma \exp[\Gamma t](N_0 - \tilde{N})$ and substitute $q = \gamma E$ for the sunk cost solution

$$q(t) = \gamma\Gamma \exp[\Gamma t](N_0 - \tilde{N}) \tag{61}$$

4.2.2 Bonds Solution

Combining (40c) and (9) provides a condition that the solution for bonds must satisfy in the long run (full derivation Appendix A.3).

$$0 = B_0 + \int_0^{\infty} e^{-rt}\left[Y - \frac{q^2}{2\gamma} - C - G\right] dt \tag{62}$$

The two terms must cancel out, which has an intuitive interpretation. The first term is the initial position of bond holdings. $B_0 > 0$ implies the country begins as a borrower, $B_0 < 0$ implies it begins as a creditor. The second term represents trade surplus if positive and deficit if negative. Therefore (62) states that if a country begins as a borrower, at some point over the time horizon it must run a trade deficit.

Linearizing the differential equation in bonds gives

$$\dot{B}(t) = \tilde{\Omega}\left[N(t) - \tilde{N}\right] - \frac{\tilde{q}}{\gamma}\left[q(t) - \tilde{q}\right] + r\left[B(t) - \tilde{B}\right] \tag{63}$$

where $\tilde{q} = 0$. Then substitute in the $N(\bar{\lambda}, t)$ solution (60) restricts the differential equation to be a linear first-order nonhomogeneous differential equation in $B(t)$

$$\dot{B}(t) = \tilde{\Omega}\left[\exp[\Gamma t](N_0 - \tilde{N})\right] + r\left[B(t) - \tilde{B}\right] \tag{64}$$

If the economy starts with bonds $B(0) = B_0$ the solution to (64) is

$$B(t) = \tilde{B} + \frac{\tilde{\Omega}}{\Gamma(\bar{\lambda}) - r}\exp[\Gamma(\bar{\lambda})t](N_0 - \tilde{N}) \tag{65}$$

where $\frac{d\dot{B}}{dN}\rceil^{\sim} = \tilde{\Omega}$ gives the effect of entry on aggregate output. And this also represents the effect of entry on the flow of bonds evaluated at steady state. $\tilde{\Omega}$ affects how accumulation of firms $N_0 \to \tilde{N}$ so $N_0 - \tilde{N} < 0$ changes stock of bonds $B(t)$. $\tilde{\Omega} > 0$ then entry strengthens home production and increases bond investment, whereas $\tilde{\Omega} < 0$ then entry weakens home production and decreases bond investment. In the Walrasian case ($\mu = 1, \nu < 1$), $\tilde{\Omega} > 0$ and the accumulation of firms leads to a reduction in bonds. The main mechanism here is that there is a positive effect of N on labor supply and output ($Y_{HN} > 0$), so that having too few firms means that wages, labor income and home production are below their steady state level. To maintain consumption, this low level of income is compensated by higher than steady state imports, financed by running down bonds. An *increase* in firms per se makes wages higher. However, the number of firms is increasing because it is below the steady-state. The stock of bonds decreases because entry implies that the initial level of N was low in the first place, not because the accumulation of firms lowers income.

However, given $\mu > 1, \nu < 1$, if μ is large enough then bonds will increase as firms are accumulated. This is because the level of profits along the path to equilibrium is large: whilst the number of firms is below equilibrium, the extra profits generated are enough to exceed the adjustment costs and lower wage. In addition, there is a capacity effect, so that productivity is higher whilst the number of firms is below equilibrium (for $\mu > 1$, free-entry leads to excessive number of firms in steady-state). In the case of $\nu \geq 1$, the flow of entry leads to an increase in the stock of bonds: this is because N has a negative effect on wages and profits, so that N below its steady state implies income above the steady state.

4.2.3 Effect of Entry on Aggregate Output

In general equilibrium, at steady state, entry may increase, decrease or have no effect on aggregate output $\tilde{\Omega} \gtreqless 0$. This depends on whether entry increases, decreases or has no effect on labor supply, which depends on whether labor is employed with decreasing, increasing or constant returns.

Proposition 3 (Entry and Aggregate Output)**.** *The effect of entry on aggregate output $\tilde{\Omega}$ is ambiguous in steady-state.*

1. *Lack of Entry:* $\tilde{\Omega} > 0 \iff 1 - \nu > \eta(\mu - 1)$
2. *Excess Entry:* $\tilde{\Omega} < 0 \iff 1 - \nu < \eta(\mu - 1)$
3. *Optimal Entry:* $\tilde{\Omega} = 0 \iff 1 - \nu = \eta(\mu - 1)$

For $\nu \geq 1$ there is always excessive entry $\tilde{\Omega} < 0$. For $\nu < 1$ all outcomes are possible.[33]

Proof.

$$\tilde{\Omega} = \left(\varepsilon - 1 + \frac{1}{\mu}\right) Y_H \tilde{h}$$

$$\tilde{\Omega} = \nu \frac{\phi}{1 - \frac{\nu}{\mu}} \left(\varepsilon - 1 + \frac{1}{\mu}\right) = \frac{\nu\phi\mu}{\mu - \nu} \left(\frac{1}{\mu} - \frac{\eta}{1 + \eta - \nu}\right)$$

$$\operatorname{sgn} \tilde{\Omega} = \operatorname{sgn} \left[\varepsilon - \left(\frac{\mu - 1}{\mu}\right)\right]$$

[33]Optimal entry refers to the number of firms that maximizes steady-state aggregate output, conditional on a markup existing. There is no maximum with perfect competition $\mu = 1$, always a lack of entry due to a positive labor effect and no negative markup (business stealing) effect.

where $\operatorname{sgn} \varepsilon = \operatorname{sgn}(1-\nu)$ since $\varepsilon = \frac{1-\nu}{1+\eta-v}$ from (32). $\square$

Next we provide a discussion of the three possible cases.[34] From the proof the outcome depends on whether the negative business stealing effect $-\left(\frac{\mu-1}{\mu}\right) \leq 0,\ \mu \in [1,\infty)$ dominates the labor elasticity to entry effect $\frac{-\eta}{1+\eta-\nu} < \varepsilon < 1$, which may be positive, negative or zero.

1. Excess Entry $\tilde{\Omega} < 0$: If there are constant $\nu = 1$ or increasing $\nu > 1$ returns to labor, $\varepsilon \leq 0$, then the fall in labor reinforces the negative business stealing effect, so there is unambiguously a negative effect of entrants on aggregate output in steady state. This is a sufficient condition but is not necessary, providing the business stealing effect is large enough it can override even a positive labor elasticity effect that arises with decreasing returns $\nu < 1$.

    (a) Example: Positive labor elasticity effect, dominated by negative business stealing effect $\nu = 0.9,\ \eta = 1$ therefore $\varepsilon = 0.\overline{09}$ with $\mu = 1.15$ business stealing is -0.13.

    (b) Constant Returns Special Case $\nu = 1$: The labor effect is zero, so only the negative business stealing effect is present. The smaller the markup $\mu \to 1$ the smaller the negative business stealing effect. But it cannot equal 1 due to the existence condition $\nu < \mu$.

    With large markups this outcome is likely. With less divisible labor $\eta \to 0$ this outcome is more likely.

2. Lack of Entry $\tilde{\Omega} > 0$: If there are decreasing returns $\nu < 1$ then $0 < \varepsilon < 1$ and the boost in labor from entry works against the negative business stealing effect, so there can be too little entry if this positive effect dominates the negative business stealing effect. $\varepsilon > 0$, hence

[34] Etro 2009; Etro and Colciago 2010 provide a discussion of 'golden rule' number of firms when there is endogenous imperfect competition, constant returns and love-of-variety. The golden rule number of firms is that which maximizes consumption and therefore output in steady-state. They show that imperfect competition causes excessive entry in steady-state, which our proposition corroborates ($\mu > 1$ and $\nu = 1$ implies $1 - \nu < \eta(\mu - 1)$, so excess entry).

$\nu < 1$, is necessary but not sufficient, sufficiency requires it is positive *and* larger than the negative business stealing effect.

(a) Example: Positive labor elasticity effect dominates negative business stealing effect $\nu = 0.9,\ \eta = 1$ therefore $\varepsilon = 0.\overline{09}$ with $\mu = 1.05$ business stealing is -0.05.

(b) Perfect Competition Special Case $\mu = 1, \nu < 1, \tilde{\Omega} > 0$: There is no negative business stealing effect, and the the existence condition $\nu < \mu$ enforces decreasing returns. Therefore entry always has a positive effect, implying lack of entry in steady state in the Walrasian (perfect competition) economy.

3. Optimal Entry $\tilde{\Omega} = 0$: A necessary condition is that the ambiguous labor elasticity effect is positive $\varepsilon > 0$, so it can counterbalance the negative business stealing effect. Therefore a necessary condition is decreasing returns $\nu < 1$.

   (a) Example: $\nu = 0.9,\ \eta = 1,\ \mu = 1.1$

4.2.4 Phase Portrait

The phase diagram in Figure 2 is analogous to the classic plot for a model of capital adjustment costs (Uribe and Schmitt-Grohé 2017, Fig. 3.2). There is a downward-sloping stable arm (saddle path) in thick black and an upward sloping unstable arm (not plotted) that intersect at steady state $(\tilde{N}, \tilde{q})$. The numerical plot captures the shape of the two nullclines. The downward sloping line represents the combinations of $\{N, q\}$ for which $\dot{q} = 0$ and the arbitrage condition is satisfied $\pi = qr$. Above the $\dot{q} = 0$ line, the arbitrage condition implies that $\dot{q} > 0$; below it implies $\dot{q} < 0$ (the arbitrary unstable green solution paths also show how the directions of motion switch either side of the nullcline). The $\dot{N} = 0$ phase line corresponds to the horizontal $N-$axis, since $\dot{N} = 0$ whenever $q = 0$. The saddle-path is downward sloping between the horizontal axis and the arbitrage line. Note that from (61) the derivative of the solution is $\dot{q} = \Gamma^2 \gamma \exp(\Gamma t)(N_0 - \tilde{N})$, so the growth (shrinkage) in the marginal cost of entry q (or net present value of a firm) is given

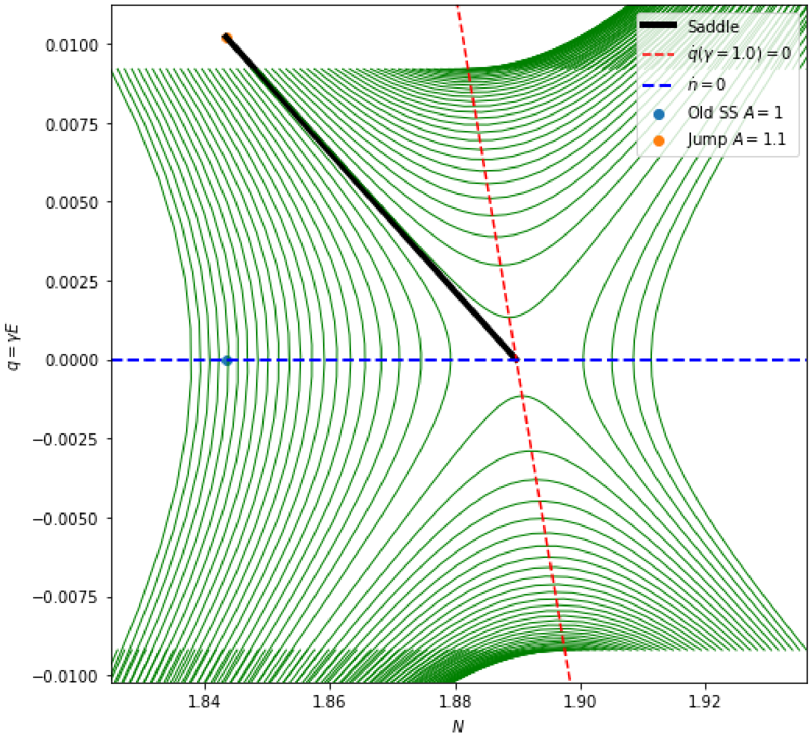

Figure 2: Phase Portrait in $\{N, q\}$ space.

in absolute terms by the stable eigenvalue

$$\left|\frac{\dot{q}}{q}\right| = \Gamma$$

with the sign being determined by whether profits are positive (firms accumulation) or negative (decumulation).

4.3 Steady-state Bonds

The linearized dynamics give an explicit solution for steady state bonds as a function of $\bar{\lambda}$ and the initial conditions N_0, B_0. Evaluate (65) at $t = 0$ implies

$$\tilde{B}(\bar{\lambda}) = B_0 - \frac{\tilde{\Omega}}{\Gamma(\tilde{N}(\bar{\lambda})) - r}(N_0 - \tilde{N}(\bar{\lambda})) \tag{66}$$

therefore the steady-state bond condition (66) and steady-state arbitrage condition (47) give the excess demand condition (49) in terms of $\bar{\lambda}$ only

$$\tilde{w}\tilde{H}(\bar{\lambda}) + r\tilde{B}(\bar{\lambda}) - \bar{C}(\bar{\lambda}) - G = 0 \tag{67}$$

We can solve this for the steady-state consumption index $\bar{\lambda}$, which then provides $\tilde{C}(\bar{\lambda}), \tilde{H}(\bar{\lambda}), \tilde{N}(\bar{\lambda})$, $\tilde{B}(\bar{\lambda})$. We cannot solve (67) analytically since it is highly nonlinear in $\bar{\lambda}$. However we can show analytically that a unique solution exists, and then solve for this numerically. A useful lemma to show uniqueness (and other results) is that the steady-state excess demand function is strictly increasing in inverse consumption, so is decreasing in consumption, under a mild regularity condition. The condition states that N_0 must begin within a given neighbourhood of $\tilde{N}$.

Lemma 4 (Excess Demand Monotonically Increasing)**.** *The steady-state market-clearing condition is monotonically increasing in* $\bar{\lambda}$

$$\tilde{w}\frac{d\tilde{H}}{d\lambda} + r\frac{d\tilde{B}}{d\bar{\lambda}} - \frac{d\tilde{C}}{d\bar{\lambda}} > 0 \tag{68}$$

if the following sufficient condition holds

$$\left(\varepsilon - 1 + \frac{1}{\mu}\right)\left(\frac{N_0}{\tilde{N}(\bar{\lambda})} - 1\right) \geq -\left(\frac{\varepsilon - 1}{\Gamma(\bar{\lambda})} + \frac{1}{r\mu}\right)(r - 2\Gamma(\bar{\lambda})) \tag{69}$$

Proof. See appendix A.7. □

The right-hand side of (69) is strictly negative and the left-hand side is ambiguous. This condition is weaker than the simpler sufficient condition $N_0 - \tilde{N}(\bar{\lambda}) \to 0$ which is commonly assumed and ensures the left-hand side is zero.[35] The condition always holds if there is entry $N_0 < \tilde{N}$ and $\varepsilon - 1 + \frac{1}{\mu} < 0$ (i.e. $\tilde{\Omega} < 0$) implying the left-hand side is positive.

Corollary 1 ($\bar{\lambda}$ Uniqueness)**.** *If* (69) *holds then there is a unique* $\bar{\lambda}$ *that solves* (68).

[35]See Turnovsky 1997, p.68 (footnote 8) for a justification of this.

Proof. Lemma 4 shows that given (69) the steady state market clearing condition (excess of income over expenditure) is strictly monotonic in $\bar{\lambda}$. Hence, if a steady-state exists it is a *unique* steady state solution for $\bar{\lambda}$. $\square$

5 Technological Change

To understand the effect of a change in technology on the economy, we outline a national income accounting framework. Total consumption consists of private and public consumption $TC = C + G$. The household budget constraint provides the national income accounting framework $Y = C + I + G + (\dot{B} - rB)$, which is equivalent to GDP. It states that aggregate output is divided between consumption, investment (in firm creation/destruction), government spending and the balance of trade (exports in the form of bond accumulation less imports in the form of payments on bonds). Investment is costs incurred to setup or close down firms $I(E) = \gamma\frac{E^2}{2}$. GNP is GDP less the balance of trade $GNP = C + I + G = Y - (\dot{B} - rB)$. In steady state $\tilde{TC} = \tilde{GNP} = \tilde{GDP} + r\tilde{B}$. In this section, since our focus is technology shocks we assume that $G = 0$. There is neither government expenditure nor taxation.[36]

5.1 Comparative Statics

An improvement in technology A reduces employment per firm but output per firm (firm scale) (12) is unaffected. Consequently an improvement in technology increases measured labor productivity and equivalently wages since $\tilde{w} = \tilde{\mathcal{P}}$.

$$\frac{d\tilde{h}}{dA} = -\frac{\tilde{h}}{\nu A} < 0 \tag{70}$$

$$\frac{d\tilde{\mathcal{P}}}{dA} = \frac{\tilde{\mathcal{P}}}{\nu A} = \frac{d\tilde{w}}{dA} = \frac{\tilde{w}}{\nu A} > 0 \tag{71}$$

[36] We have supplementary results that shows our results hold for government spending shocks.

Therefore in the long run technological progress crowds out labor at the firm-level but firm size remains constant. These comparative statics are simple as they only depend on exogenous variables. However, the aggregate endogenous variables $\{\bar{C}, \tilde{N}, \tilde{B}\}$ ((7), (47), (66)), excluding $\tilde{q}$ which is zero, are a function of A directly but also depend on $\bar{\lambda}(A)$. Therefore technology change has a direct (partial) and an indirect (consumption) effect.[37] From the steady-state market clearing condition, the implicit function theorem implies that technology unambiguously increases consumption. This rise in consumption (indirect effect) decreases aggregate labor and number of firms, whereas the direct partial effects of increased technology increase labor and number of firms. Overall, the partial effect dominates in the number of firms case, whereas it is ambiguous in the labor case. The increase in the stock of firms implies an increase in aggregate output, and a bond response that depends on the whether there are excessive, insufficient or optimal number of firms.

Proposition 4 (Long-run Effect of Technology). *A permanent improvement in technology:*

$$\frac{d\bar{C}}{dA} > 0 \tag{72}$$

$$\frac{d\tilde{N}}{dA} > 0 \tag{73}$$

$$\operatorname{sgn} \frac{d\tilde{B}}{dA} = \operatorname{sgn} -\tilde{\Omega} \tag{74}$$

$$\operatorname{sgn} \frac{d\tilde{H}}{dA} = \operatorname{sgn} \left[B_0 - \frac{\tilde{\Omega}}{\Gamma - r} N_0 \right] \tag{75}$$

$$\frac{d\tilde{Y}}{dA} = \tilde{y} \frac{d\tilde{N}}{dA} > 0 \tag{76}$$

The effect on the labor supply is ambiguous because there is a conflict of income and substitution effects: the higher wage causes a substitution effect for less leisure and more consumption, which increases labor. Whereas the income effect increases leisure and decreases labor. Which effect dominates

[37] We call the indirect effect a consumption effect as $\bar{\lambda}(A)$ is inverse consumption by (7).

depends on the level of initial wealth. From (66) $B_0 - \frac{\tilde{\Omega}}{\Gamma - r} N_0$ is the initial value of wealth in terms of bonds.[38] If $\tilde{\Omega} > 0$, that is $\nu < 1$ and μ small enough, then a sufficient condition for employment to increase $\frac{d\tilde{H}}{dA} > 0$ is that bond holdings are non-negative $B_0 \geq 0$. Likewise, if $\tilde{\Omega} < 0$, (for which $\nu \geq 1$ is sufficient) then a sufficient condition for employment to decrease $\frac{d\tilde{H}}{dA} < 0$ is that bond holdings are non-positive $B_0 \leq 0$.

Bonds respond in the opposite direction to the entry effect on output. If technology-induced entry increases GDP, then bonds decrease (less borrowing is necessary). If technology-induced entry decreases GDP, then bonds increase (more borrowing is necessary). Since steady-state bonds only depend on technology through $\tilde{N}$, the bond response follows the number of firms increase:m $\frac{d\tilde{B}}{dA} = \frac{d\tilde{B}}{dN}\frac{d\tilde{N}}{dA}$, and to a first-order approximation sgn $\frac{d\tilde{B}}{dN} \approx$ sgn $-\tilde{\Omega}$.[39] Similarly the increase in number of firms determines that aggregate output increases as long-run output per firm (firm scale) is constant.

5.2 Comparative Dynamics

From the dynamic solution for number of firms (60), we can see that on impact $t = 0$ of a shock the number of firms is fixed $N(0) = N_0$ fixed, whereas entry adjusts $E(0) = \Gamma(N_0 - \tilde{N})$, which affects the stock of firms an instance later. In other words number of firms is a stock (state) variable, and entry is a flow (jump) variable. Thus entry jumps the economy onto its stable manifold instantaneously as the shock hits, subsequently the number of firms responds as the economy evolves along this manifold. Therefore the difference between the impact and long-run effects depend on the effect of entry.

On impact the labor effect is ambiguous, as in the long run, due to competing substitution and income effects. The reason is also the same (income and substitution effects may clash). However, if we look at the difference between the impact and long-run effect, this depends on whether there is an

[38]From (66), $-\frac{\tilde{\Omega}}{\Gamma - r} N_0 = \tilde{B} - B_0 - \frac{\tilde{\Omega}}{\Gamma - r}\tilde{N}$ thus the term $-\frac{\tilde{\Omega}}{\Gamma - r} N_0$ is the present value of the bonds that would have been decumulated/accumulated if $\tilde{N} = 0$.

[39]The approximation arises from assuming we begin close to steady-state $N_0 - \tilde{N} \to 0$. From (66) removes the effect of the eigenvalue responding to $\tilde{N}$.

	$\nu < 1$	$\nu > 1$	$\nu = 1$
$B_0 > \frac{\tilde{\Omega}}{\Gamma - r} N_0$	Increase, Overshoot	Increase, Undershoot	Increase, Constant
$B_0 < \frac{\tilde{\Omega}}{\Gamma - r} N_0$	Decrease, Overshoot	Decrease, Undershoot	Decrease, Constant
$B_0 = \frac{\tilde{\Omega}}{\Gamma - r} N_0$	Constant, Overshoot	Constant, Undershoot	Constant, Constant

Table 1: Conditions for Taxonomy of Labor Dynamics

increasing or diminishing marginal product of labor at the firm level. When $\nu < 1$, on impact there is a negative relationship between the real wage and employment.; when $\nu > 1$ a positive relation; when $\nu = 1$ no relation. We can thus get undershooting of employment ($\nu > 1$) or overshooting ($\nu < 1$) on impact relative to the new long-run level depending on whether entry increases or decreases the marginal product.

Proposition 5. *Impact versus long-term effects of technological change on labor and wages.*

1.

$$\operatorname{sgn}\left[\frac{dH(0)}{dA} - \frac{dH(\infty)}{dA}\right] = \operatorname{sgn} H_N = \operatorname{sgn}\left[\nu - 1\right] \tag{77}$$

2.

$$\frac{dw(0)}{dA} = \frac{Y_{HH}}{\mu}\frac{dH(0)}{dA} + \frac{\tilde{w}}{A\nu} \tag{78}$$

$$\operatorname{sgn}\left[\frac{dw(0)}{dA} - \frac{dw(\infty)}{dA}\right] = \operatorname{sgn}\left[\nu - 1\right] \tag{79}$$

Table 1 captures the combination of static (Proposition 4) and dynamic effects (Proposition 5) on labor. Rows capture the static effect that labor might in the long-run increase, decrease or remain constant depending on initial wealth. Columns capture the dynamic effect that labor might initially overshoot, undershoot or perfectly reflect its long-run level.

5.2.1 Productivity

Lastly, we turn to the impact effect of the technological improvement on measured productivity $\mathcal{P}$.[40] First there is the pure technology effect: an increase in technology boosts output. Second there is the *capacity utilization effect*, the impact of changes in employment and output per firm relative to their long-run level. The *capacity utilization effect* is in general ambiguous in sign because $dH(0)/dA$ can take either sign. When employment increases on impact, we have *overshooting*, so that the short run impact of the technological change exceeds the long-run effect. When employment decreases on impact, we have *undershooting* of underlying measured productivity. A direct comparison of the impact and long-run effect indicates that:

Proposition 6. *Impact versus long-run productivity effects.*

$$\frac{d\mathcal{P}(0)}{dA} - \frac{d\mathcal{P}(\infty)}{dA} = (\mu - 1)\tilde{\mathcal{P}}\frac{dH(0)}{dA} \tag{80}$$

If the impact effect on employment is positive, then productivity overshoots its long-run value, if employment falls on impact, it undershoots. It is interesting to note that this result holds irrespective of whether the marginal product of labor is increasing or decreasing. If we start from a free-entry equilibrium, then AC is decreasing (since $P = AC > MC$) for all values of ν. In other words there are increasing returns from the fixed cost. A marginal improvement in technology will still leave the AC curve downward sloping at the existing level of employment. Hence, if employment decreases on impact, then you move back up the AC curve thus tending to decrease productivity. If employment rises on impact, then you move down the AC curve and increase measured productivity. Hence, it is the fact that there are locally increasing returns to employment that drives the simple relationship between employment and productivity independently of the technology parameter ν. The degree of increasing returns in free-entry equilibrium is determined by the degree of imperfect competition.

[40] Savagar and Dixon 2017 explore this endogenous productivity result in a richer model of the closed-economy with capital and strictly decreasing returns.

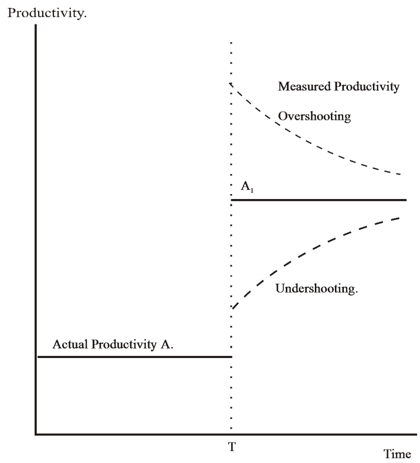

Figure 3: Undershooting and Overshooting of Measured Productivity.

The fact that capacity utilization causes endogenous productivity dynamics is important, since it implies that the time profile of measured productivity will differ from, and may tend to mask, the true changes in underlying technology. For example, if there is measured productivity overshooting, then a permanent change in technology leads to an exaggerated instantaneous impact, that dies away to the permanent change. On the other hand, if there is a capacity effect which reduces measured productivity at first (capacity widening), the measured technology increase will adjust up to the full effect. The deviations of measured productivity from technological change become small when there is near-perfect competition, and in the limiting Walrasian economy they disappear. This indicates that it may be misleading to use measured productivity as a guide to technological change in the short-run unless the economy has almost perfect competition in the output market. We show the two cases of measured productivity undershooting and overshooting in figure 3. In both cases, we have a permanent step change in technology with A rising to A_1 at time T, so that the corresponding steady states are $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{P}}_1$ respectively.

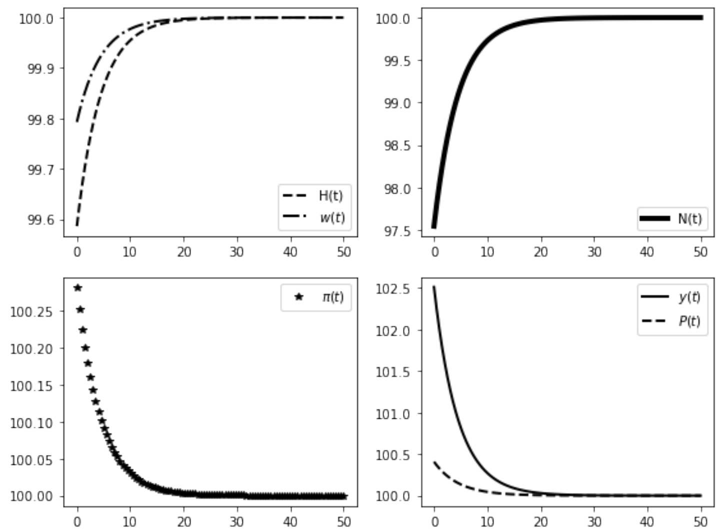

Figure 4: $A = 1 \rightarrow A = 1.1$ DRS $\nu < 1$

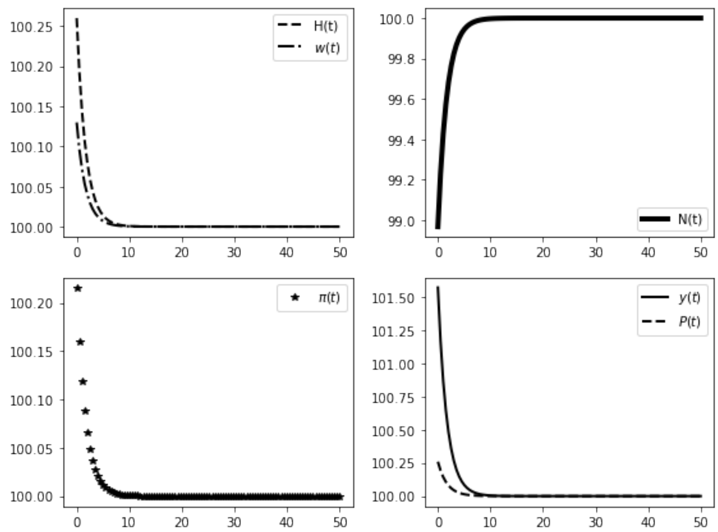

Figure 5: $A = 1 \rightarrow A = 1.1$ IRS $\nu > 1$

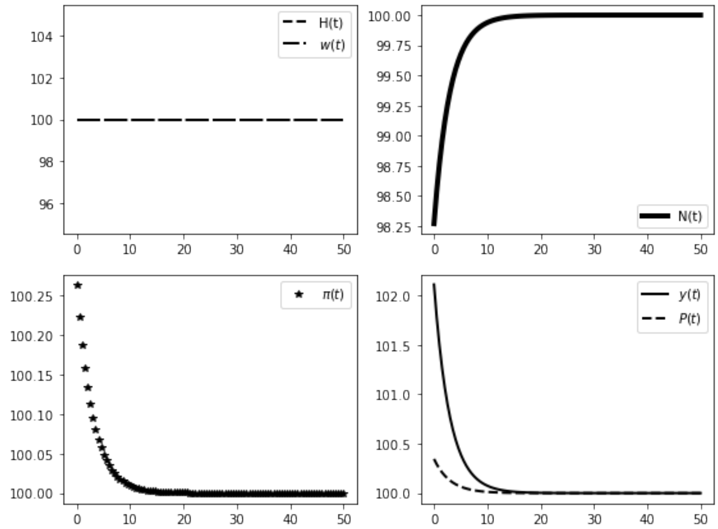

Figure 6: $A = 1 \rightarrow A = 1.1$ CRS $\nu = 1$

6 Entry Regulation Shock

The parameter γ in the cost of entry equation (36) represents a dynamic barrier to entry that we interpret as a red tape parameter. It captures how sensitive the cost of firm entry is to the number of firms that are entering in an instance. If a resource needed to setup a firm is in inelastic supply, like a government office that provides certificates to enter an industry, then the resource becomes congested as more firms seek the certification and this makes entry more costly: a firm may wait until a less congested period to attain certification. An economy/industry with lots of these red tape procedures will have a large γ, whereas an economy with less red tape will have a low γ. In this section we study a 'deregulatory' policy that corresponds to a cut in γ.[41]

Proposition 7. *The economy's speed of adjustment is monotonically decreasing in regulation of business creation.*

[41] We adopt the term deregulatory shock following Bilbiie, Ghironi, and Melitz 2007 and authors who interpret entry costs as influenced by regulation (Blanchard and Giavazzi 2003; Poschke 2010; Barseghyan and DiCecio 2011). Whereas these focus on differences in fixed exogenous sunk costs and changes in the stock of operating firms, our interest is endogenous sunk costs and changes in speed of adjustment of firms.

The magnitude of the stable root captures the economy's speed of adjustment, as it dictates the speed of adjustment of the sole state variable (number of firms) through the exponential term of (60). Taking the derivative of the stable root, which is negative, with respect to the regulatory parameter gives[42]

$$\Gamma_\gamma = \Gamma_\Delta \Delta_\gamma = \frac{\Delta_\gamma}{(r^2 - 4\Delta)^{\frac{1}{2}}} = \frac{-\Delta}{\gamma(r^2 - 4\Delta)^{\frac{1}{2}}} > 0 \tag{81}$$

The stable root is increasing in the discriminant and the discriminant $\Delta_\gamma = -\frac{\Delta}{\gamma}$ is increasing in the regulatory parameter. Therefore an increase in regulation, increases the the value of the negative root moving it closer to zero and implying slower adjustment. The effect of deregulation on speed of adjustment is striking in the simulated impulse response functions. Figure 7 is comparable to Figure 4, and steady-state is reached in approximately one-fifth of the time. The result implies that economies with less red tape recover faster following a shock.[43] In the context of labor responses to technology shocks, it implies that labor achieves its new steady state faster. This an important new dimension to the labor responses debate, because although we show that either overshooting or undershooting are plausible, this may not matter much if recovery is sufficiently fast due to nimble firms' adjustment.

More generally, the result implies that demand management could be teamed with structural reform in the form of entry deregulation. A given stimulus can have a faster effect with a less bureaucratic firm creation process. This relates to policy publications by the IMF and growing academic literature, much of which focuses on structural reform in Europe (e.g.di Mauro and Lopez-Garcia 2015).[44]

[42]This result is for a given steady-state $\tilde{N}(\bar{\lambda}))$ as γ will also affect $\tilde{N}$ through $\bar{\lambda}$.

[43]This line of analysis relates to Chatterjee 2005 who focuses on speed of convergence related to capital utilization.

[44]For example, see The Case for Fiscal Policy to Support Structural Reforms (IMF blog, 2017) and Eurozone rebalancing: Are we on the right track for growth? Insights from the CompNet micro-based data (voxEU, Bartelsman, di Mauro, Dorrucci, 2015) on the policy side and Cacciatore, Duval, et al. 2016a; Cacciatore and Fiori 2016; Cacciatore, Duval, et al. 2016b on the academic side.

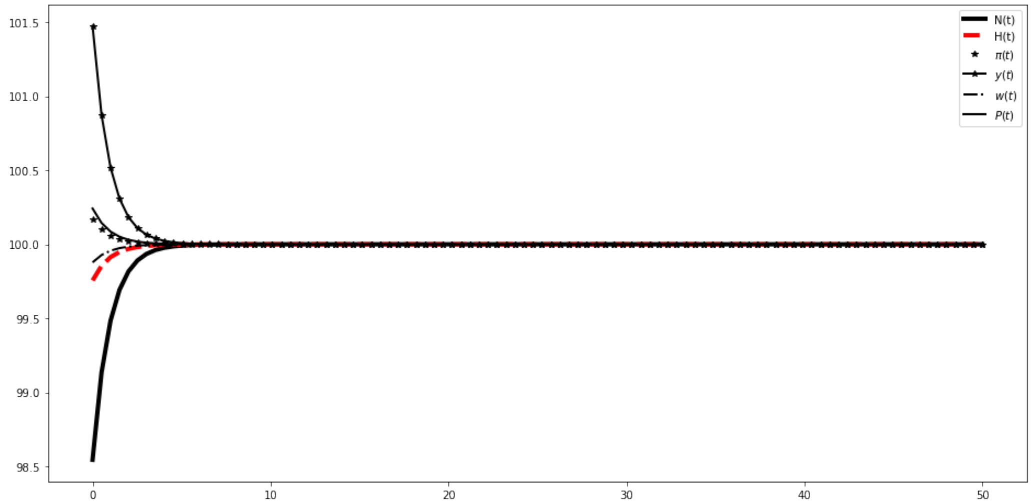

Figure 7: Deregulation $A = 1 \to A = 1.1$, $\gamma = 0.05$

7 Conclusion

This paper focuses on the effects of dynamic entry in a small open economy. We emphasize nontrivial labor market responses to firm entry when firm-level labor efficiency differs. We show that deregulation of the entry process affects speed of adjustment in the economy. A deregulatory policy, which we model as a cut in red tape, can increase the speed at which technology shocks (or fiscal stimulus) are experienced through the firm creation or destruction channel.

We provide a tractable model that provides testable implications for future applied research. The main insight is that if firm entry is slow to react, then the response of labor to technology shocks will depend on whether labor is employed with decreasing, increasing or constant returns to scale at the firm level. As returns to labor change across industries, and industries vary in significance across economies, some economies will see labor increase following a shock, whereas others will see labor decrease, and some will see little effect. Furthermore the persistence of these deviations will depend on the level of regulation, and consequently on the pace of firms' adjustment.

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A Appendix

A.1 Household Optimization

The Hamiltionian and optimality conditions are

$$\hat{\mathcal{H}}(t) = U(C, H) + \lambda(t)[rB + wH + \Pi - C - G] \tag{82}$$

$$\begin{aligned}
\hat{\mathcal{H}}_C &= 0: & U_C(C) - \lambda &= 0 & (83)\\
\hat{\mathcal{H}}_H &= 0: & U_H(H) + \lambda w &= 0 & (84)\\
\hat{\mathcal{H}}_B &= \rho\lambda - \dot{\lambda}: & \lambda r &= \rho\lambda - \dot{\lambda} & (85)\\
\hat{\mathcal{H}}_\lambda &= \dot{B}: & \dot{B} &= rB + wH + \Pi - C - G & (86)
\end{aligned}$$

The presence of a small open economy and international capital markets $\rho = r$ means that the household can completely smooth its consumption so (85) implies $\dot{\lambda} = 0$. Therefore marginal utility of wealth is unchanging over time. $\lambda = \bar{\lambda}$ combined with additively separable preferences $u_{CH} = 0$ this implies from (83) that consumption is constant and in a one-one relationship with marginal utility of wealth.[45]

$$\bar{C} = C(\bar{\lambda}) \tag{87}$$

This relationship from (83) then implies labor only varies with real wage from (84)

$$H = H(\bar{\lambda}, w) = H(\bar{C}, w) \tag{88}$$

This represents the households labor supply.

[45]We could not make the final step from (83) is $u_{CH} \neq 0$. Imposing additive separability and therefore constant consumption, we simplify analysis of dynamics as C can be treated as fixed.

A.2 General Equilibrium Effect of Entry on Output

There are two ways to think of the effect of an entrant on aggregate output $\frac{dY}{dN}$, and they offer different intuitions. The first begins with $Y = Ny$ and the second begin with $Y = AN^{1-\nu}H^{\nu} - N\phi$.

1. $\frac{dY(N,y(N,H))}{dN} = \frac{d[Ny]}{dN} = y + N\frac{dy}{dN}$ An entrant always causes 'business stealing' from other firms: a fall in output at the firm level or analogously, by (20), a fall in an each incumbents' profits.

$$\frac{dy}{dN} < 0 \tag{89}$$

$$\frac{dy}{dN} = \frac{d\left(AN^{-\nu}H^{\nu} - \phi\right)}{dN} \tag{90}$$

$$= -\nu\frac{(y+\phi)}{N} + \nu\frac{(y+\phi)}{H}\frac{dH}{dN} \tag{91}$$

$$= \nu\frac{(y+\phi)}{N}\left[\varepsilon - 1\right] < 0 \tag{92}$$

$$= Y_H\frac{h}{N}\left[\varepsilon - 1\right] \tag{93}$$

Therefore the aggregate business stealing effect is

$$N\frac{dy}{dN} = \nu(y+\phi)(\varepsilon - 1) \tag{94}$$

This also implies the effect on operating profits is negative and less than proportional

$$\frac{d\pi}{dN} = \left(1 - \frac{\nu}{\mu}\right)\frac{dy}{dN} < 0 \tag{95}$$

At the aggregate level it is not clear whether the negative business stealing effect of an entrant aggregated across all incumbents offsets the positive effect of the new firms' extra output.

$$
\begin{aligned}
\frac{dY}{dN} &= \frac{d(Ny)}{dN} && (96)\\
&= y + N\frac{dy}{dN} && (97)\\
&= y + \nu A h^{\nu}(\varepsilon - 1) && (98)\\
&= Ah^{\nu}(1 - (1-\varepsilon)\nu) - \phi && (99)\\
&= \frac{(1-\nu)(1+\eta)}{1+\eta-\nu} Ah^{\nu} - \phi && (100)\\
&= \varepsilon(1+\eta)Ah^{\nu} - \phi && (101)
\end{aligned}
$$

The final representation makes clear the crucial effect of returns to scale. It reads that an entrant has a negative effect by bringing in an extra fixed cost, but it has another positive negative or zero effect depending on ε.

2. Alternatively use (18), where the first term is the partial derivative effect of an entrant which we have explained is ambiguous based on ν, and the second term is the labor response which is also ambiguous based on ν.

$$
\begin{aligned}
\frac{dY(N,H)}{dN} &= \frac{d[AN^{1-\nu}H^{\nu} - N\phi]}{dN} = Y_N + Y_H H_N && (102)\\
&= \pi - \left(1 - \frac{1}{\mu}\right) Y_H \frac{H}{N} + Y_H H_N && (103)\\
&= \pi - \left(1 - \frac{1}{\mu} - \varepsilon\right) Y_H h && (104)
\end{aligned}
$$

A.3 Bonds

The dynamic equation (40c) is a first-order, linear, nonhomogeneous ordinary differential equation in B. Rewrite in standard form

$$
\dot{B} - rB = Y - \frac{q^2}{2\gamma} - C - G \tag{105}
$$

Multiply by the integrating factor e^{-rt}

$$e^{-rt}\dot{B} - re^{-rt}B = e^{-rt}\left[Y - \frac{q^2}{2\gamma} - C - G\right] \tag{106}$$

Notice the left-hand side as the result of a product rule differentiation, and use this to help integrate

$$e^{-rt}B = \kappa + \int_0^\infty e^{-rt}\left[Y - \frac{q^2}{2\gamma} - C - G\right]dt \tag{107}$$

To find the constant of integration κ, evaluate at $t = 0$ and use the initial condition $B(0) = B_0$

$$B(0) = \kappa = B_0 \tag{108}$$

Substitute this back in (107), then evaluate at $t \to \infty$. Use the transversality condition (9) which makes the left-hand side zero as $\lambda = \bar{\lambda}$. Therefore

$$0 = B_0 + \int_0^\infty e^{-rt}\left[Y - \frac{q^2}{2\gamma} - C - G\right]dt \tag{62}$$

A.4 Profit Maximization with Variable Returns to Scale

$$\max_h \pi_i = p_i y_i - wh_i \tag{109}$$

$$\text{s.t.} \quad \frac{p_i}{P} = \left(\frac{Y}{N^\varsigma y_i}\right)^{\frac{1}{\theta}} \tag{11}$$

$$y_i = Ah_i^\nu - \phi \tag{12}$$

$$\pi = \left(\frac{Y}{N^{\varsigma}}\right)^{\frac{1}{\theta}} (Ah^{\nu} - \phi)^{1-\frac{1}{\theta}} - wh \tag{110}$$

$$\pi_h = \left(\frac{Y}{N^{\varsigma}}\right)^{\frac{1}{\theta}} \left(1 - \frac{1}{\theta}\right) (Ah^{\nu} - \phi)^{-\frac{1}{\theta}} \cdot A\nu h^{\nu-1} - w \tag{111}$$

$$\pi_{hh} = -\frac{1}{\theta}\frac{(\pi_h + w)}{y}\frac{(y+\phi)\nu}{h} + \frac{(\pi_h + w)(\nu - 1)}{h} \tag{112}$$

$$= \frac{\pi_h + w}{h}\left[\nu\left(1 - \frac{1}{\theta} - \frac{\phi}{\theta y}\right) - 1\right] \tag{113}$$

The second-order condition $\pi_{hh} < 0$ for maximization is always satisfied when $\nu \leq 1$. However with increasing returns $\nu > 1$ it is possible that the term in square brackets is positive, unless the following necessary and sufficient condition holds $\theta\left(\frac{\theta-1}{\theta} - \frac{1}{\nu}\right) < \frac{\phi}{y}$. Throughout the paper we impose that the markup $\frac{\theta}{\theta-1} \equiv \mu$ exceeds returns to scale $\mu > \nu$ (this is necessary for a well-defined steady-state), but it is also a sufficient condition for the second-order condition to hold since $\frac{1}{\mu} - \frac{1}{\nu} < 0 < \frac{\phi}{\theta y}$. Under perfect competition $\theta \to \infty$ so $\mu \to 1$, there must be increasing marginal costs $\nu < 1$ which gives the outcome that with a fixed cost, Walrasian equilibrium only exists with increasing marginal costs, where marginal cost intersect minimum average costs at a firm's *minimum efficient scale.*

To find the profit maximizing ($\pi_h = 0$) outcome, exploit symmetry $Y/N = y$.

$$\pi_h = \left(\frac{Y}{N^{\varsigma} y}\right)^{\frac{1}{\theta}} \left(1 - \frac{1}{\theta}\right) \cdot A\nu h^{\nu-1} - w = 0 \tag{114}$$

$$\pi_h = N^{\frac{1-\varsigma}{\theta}} \left(1 - \frac{1}{\theta}\right) \cdot A\nu h^{\nu-1} - w = 0 \tag{115}$$

$$\pi_h = N^{\frac{1-\varsigma}{\theta}} \left(\frac{\theta - 1}{\theta}\right) \nu A h^{\nu-1} - w = 0 \tag{116}$$

where we ignore love of variety by assuming $\varsigma = 1$.

A.5 General Equilibrium Labor Behavior

Proof of Proposition 2. Take the derivative of (26)

$$H_N = \frac{1-\nu}{1+\eta-\nu}\frac{H}{N} \tag{117}$$

Therefore, the elasticity follows naturally

$$\varepsilon = H_N \frac{N}{H} = \frac{1-\nu}{1+\eta-\nu} \tag{118}$$

The elasticity is less than 1, it approaches 1 in the indivisible labor limit.

$$\lim_{\eta \to 0} \varepsilon = 1 \tag{119}$$

$$\lim_{\eta \to \infty} \varepsilon = \begin{cases} 0^+ & \nu < 1 \\ 0^- & \nu > 1 \end{cases} \tag{120}$$

□

A.6 Jacobian Results

In the results that follow, the trace, determinant, eigenvalue relationships are useful

$$\Delta = \Gamma\Gamma^U \tag{121}$$

$$r = \Gamma + \Gamma^U \tag{122}$$

$$\Delta = \Gamma(r - \Gamma) \tag{123}$$

$$(r^2 - 4\Delta)^{\frac{1}{2}} = r - 2\Gamma \tag{124}$$

The determinant of the entry subsystem $\det(\mathbf{B}) = \Delta(\tilde{N}(\bar{\lambda}))$ is increasing in $\bar{\lambda}$.

$$\Delta_\lambda = \Delta_N \tilde{N}_\lambda = -\frac{\Delta}{\tilde{N}} \cdot \frac{\tilde{N}}{\eta\bar{\lambda}} = -\frac{\Delta}{\eta\bar{\lambda}} > 0 \tag{125}$$

The stable root is increasing in the determinant

$$\Gamma_\Delta = -\frac{r}{2}\left(\frac{1}{2}\left(1-\frac{4\Delta}{r^2}\right)^{\frac{-1}{2}} \cdot \frac{-4}{r^2}\right) \tag{126}$$

$$= \frac{1}{(r^2-4\Delta)^{\frac{1}{2}}} = \frac{1}{r-2\Gamma} > 0 \tag{127}$$

and therefore increasing in the number of firms

$$\frac{d\Gamma}{d\tilde{N}} = \Gamma_\Delta \Delta_N = \frac{\Gamma(\Gamma-r)}{r-2\Gamma}\frac{1}{\tilde{N}} > 0 \tag{128}$$

Therefore the stable root is increasing in $\bar{\lambda}$

$$\Gamma_{\bar{\lambda}} = \Gamma_\Delta \Delta_\lambda = \Gamma_\Delta \Delta_{\tilde{N}} \tilde{N}_\lambda > 0 \tag{129}$$

which proves Lemma 3.

A.7 Steady-state Proofs

Repeating the steady-state bond condition here

$$\tilde{B}(\bar{\lambda}, A) = B_0 - \frac{\tilde{\Omega}}{\Gamma(\tilde{N}(\bar{\lambda}))-r}(N_0 - \tilde{N}(\bar{\lambda})) \tag{66}$$

The total derivative of steady-state bonds with respect to inverse consumption is

$$\frac{d\tilde{B}}{d\bar{\lambda}} = -\tilde{\Omega}\left(\frac{d\left(\frac{N_0-\tilde{N}(\bar{\lambda})}{\Gamma(N(\bar{\lambda}))-r}\right)}{d\bar{\lambda}}\right) = \tilde{\Omega}\left[\frac{(\Gamma(\bar{\lambda})-r)\frac{d\tilde{N}}{d\bar{\lambda}} + [N_0-\tilde{N}(\bar{\lambda})]\frac{d\Gamma(\tilde{N})}{d\lambda}}{(\Gamma(\bar{\lambda})-r)^2}\right] \tag{130}$$

The response of steady-state bonds to inverse consumption $\bar{\lambda}$ is ambiguous because both $\tilde{\Omega}$ and $[N_0 - \tilde{N}(\bar{\lambda})]$ are ambiguously signed. Since this model is path-dependent (steady-state depends on initial conditions $\tilde{N}(\bar{\lambda}, N_0)$ due to (66)), we cannot evaluate at $N_0 = \tilde{N}$, which removes the changing eigenvalue effect (see Caputo 2005, p. 475-477 for this common approach).[46] Instead we follow Turnovsky 1997, p.68 (footnote 8) and assume this component $[N_0 - \tilde{N}]$ is small, which – to a linear approximation – removes the changing eigenvalue effect.

Lemma 5. *The effect of a change in the consumption index on bonds is*

$$\frac{d\tilde{B}}{d\bar{\lambda}} = \frac{\tilde{\Omega}}{\Gamma(\tilde{N}(\bar{\lambda})) - r}\left[\frac{\Gamma}{r - 2\Gamma}\left(\frac{r}{\Gamma} - 3 + \frac{N_0}{\tilde{N}}\right)\right]\frac{\tilde{N}}{\bar{\lambda}\eta} \tag{131}$$

Proof. From (66) a change in consumption index only affects steady-state bonds indirectly through its effect on steady-state stock of firms

$$\frac{d\tilde{B}}{d\bar{\lambda}} = \frac{d\tilde{B}}{d\tilde{N}}\frac{d\tilde{N}}{d\bar{\lambda}} \tag{132}$$

Then steady-state stock of firms affects bonds directly $\frac{\partial \tilde{B}}{\partial \tilde{N}}$ through $\tilde{N}$ and indirectly $\frac{d\tilde{B}}{d\Gamma}\frac{d\Gamma}{d\tilde{N}}$ through the eigenvalue $\Gamma(\tilde{N}(\bar{\lambda}))$:

$$\frac{d\tilde{B}}{d\tilde{N}} = \frac{\partial \tilde{B}}{\partial \tilde{N}} + \frac{d\tilde{B}}{d\Gamma}\frac{d\Gamma}{d\tilde{N}} = \frac{\tilde{\Omega}}{\Gamma(\tilde{N}(\bar{\lambda})) - r}\left[1 + \left(\frac{N_0 - \tilde{N}(\bar{\lambda})}{\Gamma(\tilde{N}(\bar{\lambda})) - r}\right)\frac{d\Gamma}{d\tilde{N}}\right] \tag{133}$$

Therefore the effect of a change in consumption index on bonds through eigenvalues is an indirect-indirect effect.

$$\frac{d\tilde{B}}{d\bar{\lambda}} = \frac{d\tilde{B}}{d\tilde{N}}\frac{d\tilde{N}}{d\bar{\lambda}} = \left(\frac{\partial \tilde{B}}{\partial \tilde{N}} + \frac{d\tilde{B}}{d\Gamma}\frac{d\Gamma}{d\tilde{N}}\right)\frac{d\tilde{N}}{d\bar{\lambda}} \tag{134}$$

$$= \frac{\tilde{\Omega}}{\Gamma(\tilde{N}(\bar{\lambda})) - r}\left[1 + \left(\frac{N_0 - \tilde{N}(\bar{\lambda})}{\Gamma(\tilde{N}(\bar{\lambda})) - r}\right)\frac{d\Gamma}{d\tilde{N}}\right]\frac{d\tilde{N}}{d\bar{\lambda}} \tag{135}$$

Using (128) the term in square brackets simplifies

$$\frac{d\tilde{B}}{d\bar{\lambda}} = \frac{\tilde{\Omega}}{\Gamma(\tilde{N}(\bar{\lambda})) - r}\left[\frac{\Gamma}{r - 2\Gamma}\left(\frac{r}{\Gamma} - 3 + \frac{N_0}{\tilde{N}}\right)\right]\frac{d\tilde{N}}{d\bar{\lambda}} \tag{136}$$

[46] Attempting this approach here introduces another fixed point problem since changing N_0 to equal $\tilde{N}$ will in turn change $\tilde{N}$ due to path-dependency.

Therefore substituting in (157) gives (131). ∎

Corollary 2. *If* $\frac{N_0}{\tilde{N}(\bar{\lambda})} < 3 - \frac{r}{\Gamma}$ *then*

$$\operatorname{sgn} \frac{d\tilde{B}}{d\bar{\lambda}} = -\operatorname{sgn} \tilde{\Omega} \tag{137}$$

Proof. From (131) this result ensures the term in curled parenthesis is negative. ∎

Hence a sufficient condition is $\frac{N_0}{\tilde{N}} < 3$, which allows for both entry and exit $-\tilde{N} < N_0 - \tilde{N} < 2\tilde{N}$. The economic interpretation is that the initial stock of firms (market size) is greater than zero and less than three times the steady-state stock of firms. This is more general than the (commonly assumed) stronger condition that the initial condition is arbitrarily close to steady state $\frac{N_0}{\tilde{N}} \to 1$. This condition simply ensures we ignore the changing eigenvalue effect.

Corollary 3. *If* $[N_0 - \tilde{N}(\bar{\lambda})] \to 0$ *then*

$$\operatorname{sgn} \frac{d\tilde{B}}{d\bar{\lambda}} = -\operatorname{sgn} \tilde{\Omega} \tag{138}$$

Proof. From (133) as $N_0 - \tilde{N}(\bar{\lambda}) \to 0$

$$\frac{d\tilde{B}}{d\tilde{N}} \approx \frac{\partial \tilde{B}}{\partial \tilde{N}} = \frac{\tilde{\Omega}}{\Gamma(\tilde{N}(\bar{\lambda})) - r} \tag{139}$$

$$\frac{d\tilde{B}}{d\bar{\lambda}} \approx \frac{\partial \tilde{B}}{\partial \tilde{N}} \frac{d\tilde{N}}{d\bar{\lambda}} = \frac{\tilde{\Omega}}{\Gamma(\tilde{N}(\bar{\lambda})) - r} \frac{\tilde{N}}{\bar{\lambda}\eta} \tag{140}$$

∎

Lemma 6 (Steady-state Existence)**.** *By the intermediate-value theorem at least one steady-state solution exists.*

Proof of Lemma 6. Split the steady-state excess demand function into two functions: an income function $f(\bar{\lambda}) = \tilde{w}\tilde{H}(\bar{\lambda}) + rB(\bar{\lambda})$ and an expenditure function $g(\bar{\lambda}) = C(\bar{\lambda}) + G$, so we have $f(\bar{\lambda}) - g(\bar{\lambda}) = 0$. Analyze the functions for the limits of $\bar{\lambda}$. Existence follows from the functional forms for $H(\bar{\lambda}, A) = (\bar{\lambda}w)^{\frac{1}{\eta}}$ and $C(\bar{\lambda}) = \frac{1}{\bar{\lambda}}$. Also that $\tilde{B}$ is bounded in (66) since $\tilde{N}$ is bounded as it is proportional to $\tilde{H}$, which lies in $[0, 1]$. $\lim_{\lambda\to 0} H = 0$ and $\lim_{\lambda\to 0} C = \infty$ so expenditure exceeds income. $\lim_{\lambda\to\infty} H = 1$ and $\lim_{\lambda\to\infty} C = 0$, so income exceeds expenditure. Hence for at least one intermediate value of λ (67) is satisfied. □

Proof of Lemma 4. We aim to show

$$\tilde{w}\frac{d\tilde{H}}{d\lambda} + r\frac{d\tilde{B}}{d\bar{\lambda}} - \frac{d\tilde{C}}{d\bar{\lambda}} > 0 \tag{68}$$

Since $\frac{d\tilde{C}}{d\bar{\lambda}} < 0$, a sufficient condition is to show that $\tilde{w}\frac{d\tilde{H}}{d\lambda} + r\frac{d\tilde{B}}{d\lambda} > 0$. That is, we show that the positive labor effect always dominates the (potentially) negative bond effect.

$$\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}} + r\frac{d\tilde{B}}{d\bar{\lambda}} = \frac{\tilde{Y_H}}{\mu}\frac{d\tilde{H}}{d\bar{\lambda}} + r\tilde{\Omega}\left[\frac{(\Gamma - r)\frac{d\tilde{N}}{d\lambda} + [N_0 - \tilde{N}]\frac{d\Gamma}{d\bar{\lambda}}}{(\Gamma - r)^2}\right] \tag{141}$$

Substitute $\tilde{\Omega} = \left(\varepsilon - 1 + \frac{1}{\mu}\right)\tilde{Y_H}\tilde{h}$ and $\frac{d\tilde{N}}{d\lambda} = \frac{d\tilde{H}}{d\lambda}\frac{1}{\tilde{h}}$

$$\begin{aligned}
&= \left[\frac{Y_H}{\mu}\frac{d\tilde{H}}{d\bar{\lambda}}(\Gamma - r) + r\left(\varepsilon - 1 + \frac{1}{\mu}\right)Y_H\frac{d\tilde{H}}{d\bar{\lambda}}\right. \\
&\qquad \left. + \frac{r\left(\varepsilon - 1 + \frac{1}{\mu}\right)Y_H\tilde{h}(N_0 - \tilde{N})}{\Gamma - r}\frac{d\Gamma}{d\bar{\lambda}}\right]\frac{1}{\Gamma - r}
\end{aligned} \tag{142}$$

$$\begin{aligned}
&= \left[\frac{1}{\mu}(\Gamma - r) + r\left(\varepsilon - 1 + \frac{1}{\mu}\right)\right. \\
&\qquad \left. + \frac{r\left(\varepsilon - 1 + \frac{1}{\mu}\right)\tilde{h}(N_0 - \tilde{N})}{(\Gamma - r)\frac{d\tilde{H}}{d\bar{\lambda}}}\frac{d\Gamma}{d\bar{\lambda}}\right]\frac{Y_H\frac{d\tilde{H}}{d\bar{\lambda}}}{\Gamma - r}
\end{aligned} \tag{143}$$

Cancel $\frac{r}{\mu}$ and use that $\frac{d\tilde{H}}{d\bar{\lambda}} = \frac{d\tilde{N}}{d\bar{\lambda}}\tilde{h}$

$$= \left[\frac{1}{\mu}\Gamma + r\left(\varepsilon - 1\right) + \frac{r\left(\varepsilon - 1 + \frac{1}{\mu}\right)\left(N_0 - \tilde{N}\right)}{\Gamma - r}\frac{\frac{d\Gamma}{d\bar{\lambda}}}{\frac{d\tilde{N}}{d\bar{\lambda}}}\right]\frac{Y_H\frac{d\tilde{H}}{d\bar{\lambda}}}{\Gamma - r} \tag{144}$$

Remembering $\varepsilon - 1 < 0$, the first two terms are negative and the third term (the changing eigenvalue term $\frac{d\Gamma}{d\bar{\lambda}}$) is ambiguous. As with signing $\tilde{B}_{\bar{\lambda}}$, a sufficient condition to remove the problematic changing eigenvalue term is $N_0 - \tilde{N} \to 0$. Although a weaker, but messier, sufficient condition is:

$$\left(\varepsilon - 1 + \frac{1}{\mu}\right)\left(\frac{N_0}{\tilde{N}} - 1\right)\frac{\Gamma}{r - 2\Gamma} \leq -\left(\frac{\Gamma}{r\mu} + \varepsilon - 1\right) \tag{145}$$

$$\left(\varepsilon - 1 + \frac{1}{\mu}\right)\left(\frac{N_0}{\tilde{N}} - 1\right) \geq -\left(\frac{\varepsilon - 1}{\Gamma} + \frac{1}{r\mu}\right)(r - 2\Gamma) \tag{146}$$

The right-hand side is negative so this condition always holds if there is entry $N_0 < \tilde{N}$ and $\varepsilon - 1 + \frac{1}{\mu} < 0$ implying $\tilde{\Omega} < 0$. Or if there is exit $N_0 > \tilde{N}$ and and $\varepsilon - 1 + \frac{1}{\mu} > 0$ implying $\tilde{\Omega} > 0$.

□

A.8 Dynamics

Rather than defining steady-state as a function of $\tilde{h}(A), \tilde{w}(A)$ as in (47) and (48), since both depend on A and we are investigating changes in A it is useful substitute out. Repeating $\tilde{B}$, expressing dependence on A, is also useful. A only affects $\tilde{B}$ through $\tilde{N}$, which it affects directly and indirectly:

$\tilde{N}(A, \bar{\lambda}(A))$ via (147).

$$\tilde{N}(\bar{\lambda}, A) = \left(\bar{\lambda}\frac{\nu}{\mu}\right)^{\frac{1}{\eta}} A^{\frac{1+\eta}{\nu\eta}} \left(\frac{\mu-\nu}{\mu\phi}\right)^{\frac{1+\eta-\nu}{\nu\eta}} \tag{147}$$

$$\tilde{H}(\bar{\lambda}, A) = \tilde{h}(A)\tilde{N}(\bar{\lambda}, A) = \left(\bar{\lambda}\frac{\nu}{\mu}\right)^{\frac{1}{\eta}} A^{\frac{1}{\nu\eta}} \left(\frac{\mu-\nu}{\mu\phi}\right)^{\frac{1-\nu}{\nu\eta}} \tag{148}$$

$$\tilde{B}(\tilde{N}(A, \bar{\lambda}(A))) = B_0 - \frac{\tilde{\Omega}}{\Gamma(\tilde{N}(A, \bar{\lambda}(A))) - r}(N_0 - \tilde{N}(\tilde{N}(A, \bar{\lambda}(A))) \tag{66}$$

Technology change has a direct (partial) and an indirect (consumption) effect on the core endogenous model variables

$$\frac{dX}{dA} = \frac{\partial X}{\partial A} + \frac{dX}{d\bar{\lambda}}\frac{d\bar{\lambda}}{dA}, \quad X \in \{\bar{C}, \tilde{N}, \tilde{B}\} \tag{149}$$

The direct (partial) effects of A holding $\bar{\lambda}$ constant are simple to calculate. There is no partial effect on consumption, only an indirect effect.

$$\frac{\partial \bar{C}}{\partial A} = 0 \tag{150}$$

$$\frac{\partial \tilde{N}}{\partial A} = \frac{(1+\eta)\tilde{N}}{\nu\eta A} > 0 \tag{151}$$

$$\frac{\partial \tilde{B}}{\partial A} \approx \frac{\tilde{\Omega}}{\Gamma - r}\frac{\partial \tilde{N}}{\partial A} \gtreqless 0 \implies \operatorname{sgn}\frac{\partial \tilde{B}}{\partial A} = \operatorname{sgn} -\tilde{\Omega} \tag{152}$$

$$\frac{\partial \tilde{H}}{\partial A} = \frac{\tilde{H}}{\nu A\eta} > 0 \tag{153}$$

From the steady state market clearing condition (67), we can use the implicit function theorem to infer that technology decreases the marginal utility of consumption and therfore increase consumption (since through (7) consumption and marginal utility are inversely related).

Proposition 8 (Technology Effect on Steady-state Consumption)**.**

$$\frac{d\bar{\lambda}}{dA} < 0 \tag{154}$$

$$\frac{d\bar{C}}{dA} = \frac{d\bar{C}}{d\bar{\lambda}}\frac{d\bar{\lambda}}{dA} > 0 \tag{155}$$

$$\frac{d\bar{C}}{d\bar{\lambda}} = -\frac{1}{\bar{\lambda}^2} < 0 \tag{156}$$

Therefore an increase in technology increases consumption (decreases marginal utility), which, from (47) and (48), will have an indirect effect of decreasing numbers of firms and labor. This is because consumption crowds out investment in firms.

$$\frac{d\tilde{N}}{d\bar{\lambda}} = \frac{\tilde{N}}{\eta\bar{\lambda}} > 0 \tag{157}$$

$$\frac{d\tilde{B}}{d\bar{\lambda}} = \frac{d\tilde{B}}{d\tilde{N}}\frac{d\tilde{N}}{d\bar{\lambda}} \approx \frac{\tilde{\Omega}}{\Gamma - r}\frac{d\tilde{N}}{d\bar{\lambda}} \implies \operatorname{sgn}\frac{d\tilde{B}}{d\bar{\lambda}} = -\operatorname{sgn}\tilde{\Omega} \tag{158}$$

$$\frac{d\tilde{H}}{d\bar{\lambda}} = \tilde{h}\frac{d\tilde{N}}{d\bar{\lambda}} = \frac{\tilde{H}}{\eta\bar{\lambda}} > 0 \tag{159}$$

Proof of Proposition 8. The total derivative of (67) with respect to technology is

$$\frac{d\tilde{w}}{dA}\tilde{H} + \tilde{w}\left(\frac{\partial\tilde{H}}{\partial A} + \frac{d\tilde{H}}{d\bar{\lambda}}\frac{d\bar{\lambda}}{dA}\right) + r\left(\frac{\partial\tilde{B}}{\partial A} + \frac{d\tilde{B}}{d\bar{\lambda}}\frac{d\bar{\lambda}}{dA}\right) - \frac{dC}{d\bar{\lambda}}\frac{d\bar{\lambda}}{dA} = 0 \tag{160}$$

Therefore

$$\frac{d\bar{\lambda}}{dA} = -\frac{\frac{d\tilde{w}}{dA}\tilde{H} + \tilde{w}\frac{\partial\tilde{H}}{\partial A} + r\frac{\partial\tilde{B}}{\partial A}}{\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}} + r\frac{d\tilde{B}}{d\bar{\lambda}} - \frac{dC}{d\bar{\lambda}}} < 0 \tag{161}$$

The denominator is positive under sufficient condition (69) or stronger sufficient condition $N_0 - \tilde{N} \to 0$. Let's focus on the numerator

$$\frac{d\tilde{w}}{dA}\tilde{H} + \tilde{w}\frac{\partial\tilde{H}}{\partial A} + r\frac{\partial\tilde{B}}{\partial A} \tag{162}$$

which appears to be ambiguous. We shall show it is positive implying (161) is negative.

$$\frac{d\tilde{w}}{dA}\tilde{H}+\tilde{w}\frac{\partial\tilde{H}}{\partial A}+r\frac{\partial\tilde{B}}{\partial A} \tag{163}$$

$$=\frac{\tilde{w}}{\nu A}\tilde{H}+\tilde{w}\frac{\tilde{H}}{\nu A\eta}+r\frac{\tilde{\Omega}}{\Gamma-r}\frac{(1+\eta)\tilde{N}}{\nu\eta A}=\frac{1+\eta}{\nu A}\left[\frac{\tilde{w}\tilde{H}}{(1+\eta)}+\frac{\tilde{w}\tilde{H}}{(1+\eta)\eta}+r\frac{\tilde{\Omega}}{\Gamma-r}\frac{\tilde{N}}{\eta}\right] \tag{164}$$

$$=\frac{1+\eta}{\nu A}\left[\frac{\tilde{w}\tilde{H}}{\eta}+r\frac{\tilde{\Omega}}{\Gamma-r}\frac{\tilde{N}}{\eta}\right]=\frac{1+\eta}{\nu A}\left[\frac{\frac{\tilde{Y_H}}{\mu}\tilde{H}}{\eta}+r\frac{\tilde{\Omega}}{\Gamma-r}\frac{\tilde{N}}{\eta}\right] \tag{165}$$

Substitute $\tilde{\Omega}=(\varepsilon-1+\frac{1}{\mu})\tilde{Y_H}\frac{\tilde{H}}{\tilde{N}}$

$$=\frac{1+\eta}{\nu A}\left[\frac{\frac{\tilde{Y_H}}{\mu}\tilde{H}}{\eta}+r\frac{(\varepsilon-1+\frac{1}{\mu})\tilde{Y_H}\frac{\tilde{H}}{\tilde{N}}}{\Gamma-r}\frac{\tilde{N}}{\eta}\right]=\frac{(1+\eta)\tilde{Y_H}\tilde{H}}{\nu A\eta}\left[\frac{1}{\mu}+r\frac{(\varepsilon-1+\frac{1}{\mu})}{\Gamma-r}\right] \tag{166}$$

$$=\frac{(1+\eta)\tilde{Y_H}\tilde{H}}{\nu A\eta}\frac{1}{(\Gamma-r)}\left[\frac{\Gamma}{\mu}+r(\varepsilon-1)\right]=\frac{(1+\eta)\tilde{N}(\tilde{y}+\phi)}{A\eta}\frac{1}{(\Gamma-r)}\left[\frac{\Gamma}{\mu}+r(\varepsilon-1)\right]>0 \tag{167}$$

Using $\frac{\tilde{H}}{\eta\bar{\lambda}}=\frac{d\tilde{H}}{d\bar{\lambda}}$ we can show

$$=\frac{(1+\eta)\bar{\lambda}}{\nu A}\frac{\tilde{Y_H}\frac{d\tilde{H}}{d\bar{\lambda}}}{(\Gamma-r)}\left[\frac{\Gamma}{\mu}+r(\varepsilon-1)\right] \tag{168}$$

Substitute (144) (ignore changing eigenvalue effect)

$$=\frac{(1+\eta)\bar{\lambda}}{\nu A}\left(\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}}+r\frac{d\tilde{B}}{d\bar{\lambda}}\right)>0 \tag{169}$$

Therefore

$$\frac{d\bar{\lambda}}{dA}=-\frac{\frac{d\tilde{w}}{dA}\tilde{H}+\tilde{w}\frac{\partial\tilde{H}}{\partial A}+r\frac{\partial\tilde{B}}{\partial A}}{\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}}+r\frac{d\tilde{B}}{d\bar{\lambda}}-\frac{dC}{d\bar{\lambda}}}=-\frac{(1+\eta)\bar{\lambda}}{\nu A}\left(\frac{\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}}+r\frac{d\tilde{B}}{d\bar{\lambda}}}{\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}}+r\frac{d\tilde{B}}{d\bar{\lambda}}-\frac{dC}{d\bar{\lambda}}}\right)<0 \tag{170}$$

$\square$

Proof of Proposition 4. **Firms**

$$\frac{d\tilde{N}}{dA} = \frac{\partial \tilde{N}}{\partial A} + \frac{d\tilde{N}}{d\bar{\lambda}}\frac{d\bar{\lambda}}{dA} \tag{171}$$

$$= \frac{(1+\eta)}{\nu\eta A}\tilde{N} - \frac{\tilde{N}}{\bar{\lambda}\eta}\left[\frac{(1+\eta)\bar{\lambda}}{\nu A}\left(\frac{\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}} + r\frac{d\tilde{B}}{d\bar{\lambda}}}{\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}} + r\frac{d\tilde{B}}{d\bar{\lambda}} - \frac{d\bar{C}}{d\bar{\lambda}}}\right)\right] \tag{172}$$

$$= \frac{\partial \tilde{N}}{\partial A}\left[1 - \frac{\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}} + r\frac{d\tilde{B}}{d\bar{\lambda}}}{\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}} + r\frac{d\tilde{B}}{d\bar{\lambda}} - \frac{d\bar{C}}{d\bar{\lambda}}}\right] = \frac{\partial \tilde{N}}{\partial A}\left[\frac{-\frac{d\bar{C}}{d\bar{\lambda}}}{\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}} + r\frac{d\tilde{B}}{d\bar{\lambda}} - \frac{d\bar{C}}{d\bar{\lambda}}}\right] > 0 \tag{173}$$

Bonds

$$\frac{d\tilde{B}}{dA} = \frac{\partial \tilde{B}}{\partial A} + \frac{d\tilde{B}}{d\bar{\lambda}}\frac{d\bar{\lambda}}{dA} = \frac{d\tilde{B}}{d\tilde{N}}\frac{\partial \tilde{N}}{\partial A} + \frac{d\tilde{B}}{d\tilde{N}}\frac{d\tilde{N}}{d\bar{\lambda}}\frac{d\bar{\lambda}}{dA} \tag{174}$$

$$= \frac{d\tilde{B}}{d\tilde{N}}\left[\frac{\partial \tilde{N}}{\partial A} + \frac{d\tilde{N}}{d\bar{\lambda}}\frac{d\bar{\lambda}}{dA}\right] = \frac{d\tilde{B}}{d\tilde{N}}\frac{d\tilde{N}}{dA} \tag{175}$$

From (133) if $N_0 - \tilde{N} \to 0$ then $\frac{d\tilde{B}}{d\tilde{N}} = \frac{\partial \tilde{B}}{\partial \tilde{N}} + \frac{d\tilde{B}}{d\Gamma}\frac{d\Gamma}{d\tilde{N}} = \frac{\tilde{\Omega}}{\Gamma - r}\left(1 + \frac{N_0 - \tilde{N}}{\Gamma - r}\frac{d\Gamma}{d\tilde{N}}\right) \approx \frac{\tilde{\Omega}}{\Gamma - r}$ thus

$$\frac{d\tilde{B}}{dA} \approx \frac{\tilde{\Omega}}{\Gamma - r}\frac{d\tilde{N}}{dA} \gtreqless 0 \implies \operatorname{sgn}\frac{d\tilde{B}}{dA} = \operatorname{sgn} -\tilde{\Omega} \tag{176}$$

Labor:

$$\frac{d\tilde{H}}{dA} = \frac{\partial \tilde{H}}{\partial A} + \frac{d\tilde{H}}{d\bar{\lambda}}\frac{d\bar{\lambda}}{dA} = \frac{\tilde{H}}{\nu A\eta} + \frac{\tilde{H}}{\nu\bar{\lambda}}\frac{d\bar{\lambda}}{dA} = \frac{\partial \tilde{H}}{\partial A}\left[1 + \frac{\nu A}{\bar{\lambda}}\frac{d\bar{\lambda}}{dA}\right] \tag{177}$$

Substitute out (170)

$$= \frac{\partial \tilde{H}}{\partial A}\left(1 - \frac{(1+\eta)\left(\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}} + r\frac{d\tilde{B}}{d\bar{\lambda}}\right)}{\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}} + r\frac{d\tilde{B}}{d\bar{\lambda}} - \frac{d\bar{C}}{d\bar{\lambda}}}\right) \tag{178}$$

$$= \frac{\frac{\partial \tilde{H}}{\partial A}}{\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}} + r\frac{d\tilde{B}}{d\bar{\lambda}} - \frac{d\bar{C}}{d\bar{\lambda}}}\left(-\eta\left(\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}} + r\frac{d\tilde{B}}{d\bar{\lambda}}\right) - \frac{d\bar{C}}{d\bar{\lambda}}\right) \tag{179}$$

Substitute out $\frac{d\tilde{H}}{d\bar{\lambda}} = \frac{\tilde{H}}{\bar{\lambda}\eta}$, $\frac{d\tilde{B}}{d\bar{\lambda}} \approx \frac{\tilde{\Omega}}{\Gamma - r}\frac{d\tilde{N}}{d\bar{\lambda}}$ and $\frac{d\bar{C}}{d\bar{\lambda}} = -\frac{1}{\bar{\lambda}^2} = -\frac{\bar{C}}{\bar{\lambda}}$

$$= \frac{\frac{\partial \tilde{H}}{\partial A}}{\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}} + r\frac{d\tilde{B}}{d\bar{\lambda}} - \frac{d\bar{C}}{d\bar{\lambda}}} \frac{1}{\bar{\lambda}} \left(\bar{C} - \tilde{w}\tilde{H} - r\frac{\tilde{\Omega}}{\Gamma - r}\tilde{N} \right) \tag{180}$$

In steady state $\tilde{C} - \tilde{w}\tilde{H} = r\tilde{B}$

$$\frac{d\tilde{H}}{dA} = \frac{\frac{\partial \tilde{H}}{\partial A}}{\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}} + r\frac{d\tilde{B}}{d\bar{\lambda}} - \frac{d\bar{C}}{d\bar{\lambda}}} \frac{1}{\bar{\lambda}} \left(r\tilde{B} - r\frac{\tilde{\Omega}}{\Gamma - r}\tilde{N} \right)$$

From (66) $\tilde{B} - \frac{\tilde{\Omega}}{\Gamma - r}\tilde{N} = B_0 - \frac{\tilde{\Omega}}{\Gamma - r}N_0$

$$\frac{d\tilde{H}}{dA} = \frac{\frac{\partial \tilde{H}}{\partial A}}{\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}} + r\frac{d\tilde{B}}{d\bar{\lambda}} - \frac{d\bar{C}}{d\bar{\lambda}}} \frac{r}{\bar{\lambda}} \left(B_0 - \frac{\tilde{\Omega}}{\Gamma - r}N_0 \right)$$

□

Proof of Proposition 5. **Labor**: Totally differentiating $H = H(\bar{\lambda}, N, A)$ keeping N fixed yields.

$$\frac{dH(0)}{dA} = \frac{dH}{d\bar{\lambda}}\frac{d\bar{\lambda}}{dA} + \frac{\partial H}{\partial A} \tag{181}$$

$$= -\frac{\partial H}{\partial A}\left[\frac{(1 + \eta - \nu)(w\frac{dH}{d\bar{\lambda}} + r\frac{dB}{d\bar{\lambda}}) - \nu\frac{dC}{d\bar{\lambda}})}{\nu\left(w\frac{dH}{d\bar{\lambda}} + r\frac{dB}{d\bar{\lambda}} - \frac{dC}{d\bar{\lambda}}\right)} \right] \tag{182}$$

As in the long-run case, the income and substitution effects of a technological improvement work in opposite directions. The difference between the long-run and impact multiplier is accounted for by the effect of entry, so that

$$\frac{dH(0)}{dA} - \frac{dH(\infty)}{dA} = \frac{dH}{dN}\frac{dN}{dA} = \frac{dH}{dN}\left[\frac{\partial N}{\partial A} + \frac{dN}{d\bar{\lambda}}\frac{d\bar{\lambda}}{dA} \right] \tag{183}$$

$$= \frac{dH}{dN}\frac{\partial \tilde{N}}{\partial A}\left[\frac{-\frac{d\bar{C}}{d\bar{\lambda}}}{\tilde{w}\frac{d\tilde{H}}{d\bar{\lambda}} + r\frac{d\tilde{B}}{d\bar{\lambda}} - \frac{dC}{d\bar{\lambda}}} \right] \tag{184}$$

$$\text{sgn}\left[\frac{dH(\infty)}{dA} - \frac{dH(0)}{dA} \right] = \text{sgn}\ \ H_N = \text{sgn}\left[1 - \nu\right]$$

Wages:

$$\frac{dw(0)}{dA} = \frac{1}{\mu}Y_{HH}\frac{dH(0)}{dA} + \frac{w}{A\nu} \tag{185}$$

Hence

$$\frac{dw(0)}{dA} - \frac{dw(\infty)}{dA} = \frac{1}{\mu}Y_{HH}\frac{dH(0)}{dA} \tag{186}$$

$$\operatorname{sgn}\left[\frac{dw(0)}{dA} - \frac{dw(\infty)}{dA}\right] = \operatorname{sgn}\left[\nu - 1\right] \tag{187}$$

The difference between the long-run and short run wage effect depends on whether an increase in employment increases the MPL ($\nu > 1, Y_{HH} > 0$), or decreases it ($\nu < 1$, $Y_{HH} < 0$). □

Proof of Proposition 6.

$$\frac{d\mathcal{P}(0)}{dA} = \frac{1}{\mu}\left(\frac{N}{H}\right)^{(1-\nu)} + (\nu-1)\frac{A}{H}\left[\frac{N}{H}\right]^{1-\nu}\frac{dH(0)}{dA} + \left[\frac{N}{H}\right]\frac{\phi}{H}\frac{dH(0)}{dA} \tag{188}$$

$$= \frac{\tilde{\mathcal{P}}}{\nu A} + (\mu-1)\tilde{\mathcal{P}}\frac{dH(0)}{dA} = \frac{d\tilde{\mathcal{P}}}{dA} + (\mu-1)\tilde{\mathcal{P}}\frac{dH(0)}{dA} \tag{189}$$

□